GROS TOPOI AS PARTIALLY LAX LIMITS OF PETIT TOPOI

FABIO NEUGEBAUER, QI ZHU

Abstract. We prove that the sheaf ∞-topos associated to a geometric site in the sense of Lurie can be written as a partially lax limit of smaller sheaf ∞-topoi. This is thus a formalization of Lurie's vision for fractured ∞-topoi to axiomatize the relation between gros and petit topoi. As a consequence, we realize the ∞-topos of κ-small condensed anima as a partially lax limit of sheaf ∞-topoi over extremally disconnected spaces, marked at the open embeddings. Moreover, we deduce a gros version of the étale exodromy theorem.

Contents

1. Introduction 1
2. Recollection on Geometric Sites 4
3. On Partially Lax Limits 4
4. Presheaf Categories as Partially Lax Limits 8
5. Sheaf Categories as Partially Lax Limits 11
6. Gros étale exodromy 13
References 14

1. Introduction

Geometric information is often organized in a global or in a local way. One virtue of this principle in topos theory is realized by so-called *gros* or *petit* topoi – in the example of topological spaces Top this is the distinction between sheaves on all (small) topological spaces $\mathrm{Sh}(\mathrm{Top})$ and sheaves $\mathrm{Sh}(X)$ on specific spaces X. On the other hand, a global object is ideally an amalgamation of its local information along with suitable gluing information: a sheaf $\mathcal{F} \in \mathrm{Sh}(\mathrm{Top})$ should be recoverable from a collection of sheaves $\{\mathcal{F}_X \in \mathrm{Sh}(X)\}_{X \in \mathrm{Top}}$ together with certain compatibilities.

Our main theorem is a formalization of this intuition, for which we use Lurie's notion of geometric sites [Lur18, Section 20]. It consists of a site $(\mathcal{G}, \tau)$ together with a family of admissible morphisms $\mathcal{G}^{\mathrm{ad}} \subset \mathcal{G}$ and suitable compatibilities, see Definition 2.1. In this language, the open embeddings on Top form a class of admissible morphisms, making our introductory example an instance of this.

Theorem A (Theorem 5.2)**.** *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site and $\mathcal{C}$ a presentably symmetric monoidal ∞-category. There is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Sh}(\mathcal{G}; \mathcal{C}) \xrightarrow{\simeq} \operatorname*{laxlim}^{\dagger}_{X \in \mathcal{G}^{\mathrm{op}}} \mathrm{Sh}\left(\mathcal{G}^{\mathrm{ad}}_{/X}; \mathcal{C}\right)$$

describing the ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{G}$ as a partially lax limit indexed by the marked ∞-category $(\mathcal{G}^{\mathrm{ad}})^{\mathrm{op}} \subset \mathcal{G}^{\mathrm{op}}$.

The monoidal structure is the sheafification of the pointwise monoidal structure. We work with symmetric monoidal ∞-categories but the arguments are equally valid for presentably $\mathcal{O}$-monoidal categories for any unital ∞-operad $\mathcal{O}$. Moreover, Theorem 5.2 generalizes to hypersheaf and module categories, cf. Remark 5.3 and Remark 5.4.

Let us informally explain the functoriality of $\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/-}; \mathcal{C})$ defining the partially lax limit. Let us denote by $\mathcal{P}^{\mathcal{C}}(-)$ the $\mathcal{C}$-valued presheaves. Pullback along a map $f\colon X \to Y$ in $\mathcal{G}$ yields a morphism of sites $f^*\colon \mathcal{G}^{\mathrm{ad}}_{/Y} \to \mathcal{G}^{\mathrm{ad}}_{/X}$. Left Kan extension followed by sheafification yields

$$\mathrm{Sh}\left(\mathcal{G}^{\mathrm{ad}}_{/Y}; \mathcal{C}\right) \hookrightarrow \mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/Y}\right) \xrightarrow{\mathrm{Lan}_{(f^*)^{\mathrm{op}}}} \mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right) \xrightarrow{L_\tau} \mathrm{Sh}\left(\mathcal{G}^{\mathrm{ad}}_{/X}; \mathcal{C}\right),$$

Date: October 3, 2026.

We will construct this functor rigorously in Section 4 and Section 5 with all the desired coherences using $(\infty,2)$-category theory. The functor $\mathrm{Sh}(\mathcal{G};\mathcal{C}) \to \mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/X};\mathcal{C})$ appearing on the X-factor of Theorem A is given by restriction along the forgetful functor $\mathcal{G}^{\mathrm{ad}}_{/X} \to \mathcal{G}$.

Our Theorem A is obtained by sheafifying the following statement about presheaf categories, denoted $\mathcal{P}^{\mathcal{C}}(-) \coloneqq \mathrm{Fun}((-)^{\mathrm{op}},\mathcal{C})$.

Theorem B (Theorem 4.4)**.** *Let $\mathcal{G}^{\mathrm{ad}} \subset \mathcal{G}$ be a pullback-stable left-cancellative wide subcategory of a small ∞-category $\mathcal{G}$. Let $\mathcal{C}$ be a cocomplete symmetric monoidal ∞-category such that the monoidal structure commutes with colimits in each variable. There is a preferred symmetric monoidal equivalence of ∞-categories*

$$\mathcal{P}^{\mathcal{C}}(\mathcal{G}) \xrightarrow{\simeq} \operatorname*{laxlim}^{\dagger}_{X\in\mathcal{G}^{\mathrm{op}}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}), \quad \Phi \mapsto \left((\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}} \xrightarrow{\mathrm{fgt}} \mathcal{G}^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}\right)_{X\in\mathcal{G}^{\mathrm{op}}}$$

where we mark by the morphisms in $\mathcal{G}^{\mathrm{ad}}$.

Fractured Topoi and Partially Lax Limits. Let us now informally recall some of the terminology used in Theorem A. We will give a more rigorous recollection in Section 2 and Section 3.

Lurie suggested a formalization of the philosophy of gros and petit topoi via so-called *fractured* ∞-*topoi* [Lur18, Section 20], which roughly is a subcategory $\mathcal{X}^{\mathrm{corp}}$ of an ∞-topos $\mathcal{X}$ satisfying various compatibilities. While sheaves on sites present ∞-topoi, the analogous notion to present fractured ∞-topoi is through *geometric sites* – meaning a site $(\mathcal{G},\tau)$ together with a wide subcategory $\mathcal{G}^{\mathrm{ad}}$ of so-called *admissible morphisms* that is compatible with the topology τ in a suitable way (Definition 2.1). By [Lur18, Theorem 20.6.3.4], the subcategory $\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}) \subseteq \mathrm{Sh}(\mathcal{G})$ yields a fractured ∞-topos, where $\mathrm{Sh}(\mathcal{G})$ plays the role of the gros topos and $\{\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/X})\}_{X\in\mathcal{G}}$ plays the role of the petit topoi. Thus, our main theorem describes the fractured ∞-topos $\mathrm{Sh}(\mathcal{G})$ as a partially lax limit over the respective petit topoi.

Partially lax limits, introduced by Gepner–Haugseng–Nikolaus [GHN17], formalize the sense in which one takes a compatible family of objects in the petit topoi. Specializing the rigorous definition from Section 3 to our Theorem A describes a sheaf $\mathcal{F} \in \mathrm{Sh}(\mathcal{G};\mathcal{C})$ equivalently by the data:

- a sheaf $\mathcal{F}_X \in \mathrm{Sh}\left(\mathcal{G}^{\mathrm{ad}}_{/X};\mathcal{C}\right)$ for each $X \in \mathcal{G}$,
- a family of compatible morphisms $f_\alpha \colon \alpha^* \mathcal{F}_Y \to \mathcal{F}_X$ for every $\alpha \colon X \to Y$ in $\mathcal{G}$, where f_α is an equivalence if α is admissible.

Moreover, this equivalence is compatible with the pointwise symmetric monoidal structure on the partially lax limit (Remark 3.6).

Simplification of the Partially Lax Limit on Certain Functors. It turns out that there are a number of functors which send partially lax limits to limits – we will recall some in Section 3.2. We apply this in the situation of Theorem A and gain extra mileage when $\mathcal{G}$ admits a terminal object $*$ so that $\lim_{\mathcal{G}^{\mathrm{op}}}$ is computed by evaluation at $*$. In this case, the restriction functor

$$\mathrm{Sh}(\mathcal{G};\mathcal{C}) \longrightarrow \mathrm{Sh}\left(\mathcal{G}^{\mathrm{ad}}_{/*};\mathcal{C}\right) \tag{1}$$

induces an equivalence on

(1) the full subcategories of dualizable objects,
(2) Picard spectra, Picard spaces and Picard groups, and
(3) rigidifications in the sense of Gaitsgory–Rozenblyum [GR17, HSSS21, Ram26],

see Corollary 5.5. While these applications admit a more elementary proof via the left adjoint of Equation (1), the partially lax limit perspective motivated us to expect these results in the first place.

Condensed Objects as a Partially Lax Limit. We want to single out an example of fractured ∞-topoi from condensed mathematics, while more classical examples are recalled in Section 2.

Fix an uncountable strong limit cardinal κ. Let ExtrDisc denote the 1-category of κ-small extremally disconnected compact Hausdorff spaces, i.e. those κ-small compact Hausdorff spaces in which the closure of an open set is open again.[1] Equivalently, these are the projective objects in the category of compact Hausdorff spaces CHaus by a theorem of Gleason [Gle58, Theorem 2.5]. We endow it with the Grothendieck topology of finitely jointly surjective families of maps. Let $\mathcal{C}$ be a cocomplete ∞-category,

[1] Let us recall that every extremally disconnected compact Hausdorff space is a profinite set.

then Clausen–Scholze resp. Barwick–Haine [BH19, Sch26b, Sch26a] introduced the *condensed objects valued in* $\mathcal{C}$ as the ∞-category of $\mathcal{C}$-valued sheaves on ExtrDisc, i.e.

$$\mathrm{Cond}(\mathcal{C}) \coloneqq \mathrm{Sh}(\mathrm{ExtrDisc};\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{ExtrDisc}^{\mathrm{op}};\mathcal{C}).$$

In work of the second-named author with Rasekh, we identified $\mathrm{Cond}(\mathrm{An})$ as a fractured ∞-topos presented by a geometric site and computed the associated petit topoi [RZ26, Theorems A and B]. Combining this with Theorem A we deduce:

Theorem C (Corollary 5.6)**.** *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Then, there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Cond}(\mathcal{C}) \simeq \operatorname*{laxlim}^{\dagger}_{X\in\mathrm{ExtrDisc}^{\mathrm{op}}} \mathrm{Sh}(X;\mathcal{C}),$$

where the open embeddings are marked.

The appeal of Theorem C is that it identifies $\mathrm{Cond}(\mathrm{Sp})$ as a partially lax limit of "extremely[2] nice" categories. Indeed, for any $X \in \mathrm{ExtrDisc}$ the symmetric monoidal category $\mathrm{Sh}(X;\mathrm{Sp})$ is compactly generated [Har25, Proposition 3.1] and rigid [HSSS21, Proposition 2.20], i.e. dualizable, objects of $\mathrm{Sh}(X;\mathrm{Sp})$ are precisely the compact objects. Moreover, equivalences in $\mathrm{Sh}(X;\mathrm{Sp})$ are detected on the points of X.

Gros Étale Exodromy. Van Dobben de Bruyn, building on work of Barwick–Glasman–Haine and Wolf [BGH20, Wol22], proves an exodromy correspondence for Postnikov complete étale sheaves: This is a way to reconstruct sheaves on a scheme as continuous functors (see Section 6) from a so-called Galois category [vDdB26, Theorem 2].

Here, the *condensed Galois ∞-category* of X from [BGH20, HHL+25, Mai26, vDdB26] is

$$\mathrm{Gal}(X)\colon \mathrm{ExtrDisc}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty},\ S \mapsto \mathrm{Fun}^{*}_{\mathrm{loc\,coh}}(\mathrm{Sh}(X_{\text{ét}}),\mathrm{Sh}(S)),$$

which for $S \in \mathrm{ExtrDisc}^{\mathrm{op}}$ consists of locally coherent geometric morphisms $\mathrm{Sh}(S) \to \mathrm{Sh}(X_{\text{ét}})$, see [vDdB26, 2.1.6]. This is functorial in X and our theorem concerns the gros version

$$\mathrm{GAL}\colon \mathrm{ExtrDisc}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty}, \qquad S \mapsto \operatorname*{laxcolim}^{\dagger}_{X\in\mathrm{Sch}} \mathrm{Gal}(X)(S),$$

where we mark the étale maps and run over κ-small qcqs schemes, see Definition 6.3. Moreover, let $\underline{\mathcal{C}} = \mathrm{Sh}(-;\mathcal{C})\colon \mathrm{ExtrDisc}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ be the condensed ∞-category associated to $\mathcal{C}$. Then, we prove a gros version of the étale exodromy theorem:

Theorem D (Theorem 6.5)**.** *Let $\mathcal{C}$ be a compactly assembled presentable ∞-category. Then, there is a preferred étale exodromy equivalence*

$$\mathrm{Sh}^{\mathrm{post}}_{\acute{e}t}(\mathrm{Sch};\mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{GAL},\underline{\mathcal{C}})$$

of ∞-categories.

The same result holds over more general base schemes T, see Remark 6.6.

Conventions. We will freely use the language of $(\infty,2)$-categories, for which we recommend [BB25, Appendix A] as a reference.

AI Disclosure. We have used LLMs to search the literature, help proofread and improve parts of the language. The mathematical content is entirely our own.

Acknowledgements. We thank Thomas Blom and Phil Pützstück for insightful conversations about partially lax limits, Remy van Dobben de Bruyn for answering questions about exodromy and Markus Hausmann for giving us feedback on a draft. FN is supported by the European Research Council (ERC) under Horizon Europe (Starting Grant BorSym, ID: 101163408) and furthermore thanks the Max Planck Institute for Mathematics in Bonn for its hospitality. QZ is grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support.

[2]...maybe we should say extremally!

2. Recollection on Geometric Sites

Lurie's fractured ∞-topoi [Lur18, Section 20] are usually presented by a geometric site. For the purpose of this article, we will only need the notion of geometric sites. Let us recall Lurie's definition.

Definition 2.1 ([Lur18, Definition 20.2.1.1, 20.6.2.1])**.** Let $\mathcal{G}$ be a small ∞-category.

(1) A *class of admissible morphisms* is a class of morphisms in $\mathcal{G}$ that is a full and replete subcategory $\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}) \subseteq \mathrm{Ar}(\mathcal{G})$ of the arrow category such that:
   a) For any object $X \in \mathcal{G}$ the identity morphism id_X is admissible.
   b) For any admissible morphism $a\colon A \to B$ and any morphism $f\colon X \to B$ in $\mathcal{G}$ the pullback $a'\colon A \times_B X \to X$ exists and is admissible.
   c) Let f, g be composable morphisms in $\mathcal{G}$. If g is admissible, then $g \circ f$ is admissible if and only if f is admissible.
   d) The subcategory $\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}) \subseteq \mathrm{Ar}(\mathcal{G})$ is closed under retracts.

   We denote by $\mathcal{G}^{\mathrm{ad}} \subset \mathcal{G}$ the wide subcategory spanned by the admissibles.
(2) A *geometric site* is a site $(\mathcal{G}, \tau)$ together with a class of admissible morphisms such that any τ-cover admits a τ-subcover generated by a set of admissible morphisms.

Remark 2.2. In this article we will *never* need condition d). It is only relevant to obtain the full strength of Lurie's fractured ∞-topoi from [Lur18, Section 20], which will play no further role in this article.

One can pull back the topology τ to τ^{ad} on $\mathcal{G}^{\mathrm{ad}}$ by [Lur18, Proposition 20.6.1.1]. Then, $(\mathrm{Sh}(\mathcal{G}), \mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}))$ forms a fractured ∞-topos in the sense of Lurie [Lur18, Theorem 20.6.3.4]. This formalizes the notion of gros and petit topoi by letting $\mathrm{Sh}(\mathcal{G})$ take the role of a gros topos and the slices $\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/X})$ for $X \in \mathcal{G}$ take the role of petit topoi.

To deal with size issues, fix an uncountable strong limit cardinal κ.

Example 2.3. The ∞-category of qcqs κ-small[3] schemes or spectral schemes with the étale/Zariski topology together with the étale morphisms/open immersions forms a geometric site [Lur18, 20.6.4].

Example 2.4. Consider the (1-)category Mfld^r of r-times differentiable (second countable, Hausdorff) κ-small manifolds with r-times differentiable maps together with a topology given by jointly surjective open embeddings. Together with the r-times differentiable open embeddings $\mathrm{Mfld}^r_{\text{ét}}$, we obtain a geometric site [Clo25, Lemma 2.4]. Let $\mathcal{C}$ be an ∞-category. The slices can be computed as $\mathrm{Sh}(\mathrm{Mfld}^r_{\text{ét}/M}; \mathcal{C}) \simeq \mathrm{Sh}(M; \mathcal{C})$ for $M \in \mathrm{Mfld}^r$, see [Clo25, Remark 2.6].

Example 2.5. Let ExtrDisc denote the (1-)category of κ-small[4] extremally disconnected compact Hausdorff spaces, i.e. those κ-small compact Hausdorff spaces in which the closure of an open set is open again. Consider a Grothendieck topology τ given by the finitely jointly surjective families of maps. Let $\mathcal{C}$ be an ∞-category. Then, Clausen–Scholze's ∞-category of *condensed objects with values in* $\mathcal{C}$ is

$$\mathrm{Cond}(\mathcal{C}) \coloneqq \mathrm{Sh}(\mathrm{ExtrDisc}; \mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{ExtrDisc}^{\mathrm{op}}, \mathcal{C}),$$

see [Sch26a, Definition 11.7]. For example, a *condensed* ∞*-category* is a condensed object valued in Cat_∞.

Let $\mathrm{ExtrDisc}^{\mathrm{open}}$ denote the wide subcategory of open embeddings. Then, $(\mathrm{ExtrDisc}, \mathrm{ExtrDisc}^{\mathrm{open}}, \tau)$ defines a geometric site by [RZ26, Theorem A]. Furthermore, [RZ26, Theorem B] computes the slices as $\mathrm{Sh}(\mathrm{ExtrDisc}^{\mathrm{open}}_{/S}; \mathcal{C}) \simeq \mathrm{Sh}(S; \mathcal{C})$ for $S \in \mathrm{ExtrDisc}$.

The main result (Theorem A) of this article relates $\mathrm{Cond}(\mathcal{C})$ and $\mathrm{Sh}(S; \mathcal{C})$ via partially lax limits.

3. On Partially Lax Limits

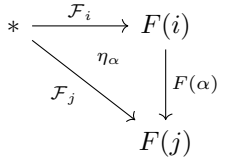
3.1. **Recollection.** Let us now give a formal version of the informal recollection of partially lax limits from Section 1. We recommend [GLP26, Section 2] for further background. We denote by Cat_∞ the ∞-category[5] of small ∞-categories, but the analogous constructions hold for large ∞-categories by passing to a larger universe.

[3] In the sense of [vDdB26, Definition 1.1.7]. The precise definition won't be important to us, but a size bound will be needed in the gros exodromy theorem (Theorem 6.5), since [vDdB26, Theorem 3.2.6] requires such a size condition.

[4] Alternatively, one may take the pyknotic perspective from [BH19] to deal with the set theory.

[5] To apply the lax natural transformation machine, we view Cat_∞ as an $(\infty, 2)$-category via the self-enrichment coming from the cartesian closedness.

Let $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ be a functor and let $*$ denote the terminal ∞-category. Informally, a point in its lax limit consists of functors $F_i\colon * \to F(i)$ for every $i \in \mathcal{I}$, together with natural transformations



$$\eta_\alpha\colon F(\alpha) \circ F_i \Longrightarrow F_j$$

for every morphism $\alpha\colon i \to j$ in $\mathcal{I}$, satisfying certain coherences. Equivalently, $\mathcal{F}_i$ is an object of $F(i)$ and $\eta_\alpha\colon F(\alpha)(\mathcal{F}_i) \to \mathcal{F}_j$ gives its transition morphism in $F(j)$. These data form a lax cone over F, i.e. a lax natural transformation $\Delta(*) \overset{\mathrm{lax}}{\Longrightarrow} F$ from the constant functor $\Delta(*)$ at the terminal category $*$ to F. Reversing the natural transformations gives an oplax cone.

Definition 3.1. The *lax limit* and *oplax limit* of $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ are the ∞-categories of lax resp. oplax cones:

$$\operatorname*{laxlim}_{\mathcal{I}} F \coloneqq \mathrm{Nat}^{\mathrm{lax}}(\Delta(*), F) \quad \text{and} \quad \operatorname*{oplaxlim}_{\mathcal{I}} F \coloneqq \mathrm{Nat}^{\mathrm{oplax}}(\Delta(*), F),$$

where $\Delta(*)$ denotes the constant functor with value the terminal ∞-category. Similarly, the *(op-)lax colimits* are given by *(op-)lax cocones*.

By straightening/unstraightening for lax natural transformations, these descriptions identify with categories of sections, see [GLP26, Remark 2.7]. In particular,

$$\operatorname*{laxlim}_{\mathcal{I}} F \simeq \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}} F),$$

where $\mathrm{Un}^{\mathrm{co}} F \to \mathcal{I}$ is the coCartesian unstraightening of F. Under this equivalence, η_α is an equivalence precisely when the corresponding section sends α to a coCartesian edge. Similarly, the oplax limit is the ∞-category of sections of the Cartesian unstraightening $\mathrm{Un}^{\mathrm{ct}} F \to \mathcal{I}^{\mathrm{op}}$.

Definition 3.2. A *marked ∞-category* is a pair $\mathcal{I}^\dagger = (\mathcal{I}, \mathcal{W})$ consisting of an ∞-category $\mathcal{I}$ together with a replete subcategory $\mathcal{W} \subseteq \mathcal{I}$ containing all objects. For two marked ∞-categories $\mathcal{C}, \mathcal{D}$, we denote by $\mathrm{Fun}^\dagger(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory of functors preserving marked edges.

The *partially (op-)lax limits* are the full subcategories

$$\operatorname*{laxlim}_{\mathcal{I}^\dagger}{}^{\dagger} F \subseteq \operatorname*{laxlim}_{\mathcal{I}} F \quad \text{and} \quad \operatorname*{oplaxlim}_{\mathcal{I}^\dagger}{}^{\dagger} F \subseteq \operatorname*{oplaxlim}_{\mathcal{I}} F$$

spanned by those (op-)lax cones whose structure transformations η_α are equivalences for all $\alpha \in \mathcal{W}$, i.e. whose restriction to $\mathcal{W}$ is strictly natural. Similarly, the *partially (op-)lax colimit* is characterized by the universal (op-)lax cocone whose restriction to $\mathcal{W}$ is strictly natural.

Example 3.3. The limit of F is the full subcategory

$$\lim_{\mathcal{I}} F \;\subseteq\; \operatorname*{laxlim}_{\mathcal{I}^\dagger}{}^{\dagger} F \;\subseteq\; \operatorname*{laxlim}_{\mathcal{I}} F$$

spanned by those lax cones for which η_α is an equivalence for every morphism α in $\mathcal{I}$. Equivalently, it consists of the coCartesian sections of $\mathrm{Un}^{\mathrm{co}} F \to \mathcal{I}$. Thus, marking all morphisms recovers the limit, while marking only equivalences recovers the lax limit. Partially lax limits interpolate between these two constructions.

Remark 3.4. Let $\mathcal{I}^\dagger$ be a marked ∞-category and $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ a functor. We write $|-|$ for the localization at the marked arrows and give the slice categories the markings induced by their forgetful functors to $\mathcal{I}$. The end and coend formulas of [Ber24, Definition 3.3 and Remark 3.4] with the conventions of [LNP25, GLP26] give

$$\operatorname*{laxlim}_{\mathcal{I}^\dagger}{}^{\dagger} F \simeq \lim_{(i\to j)\in \mathrm{Tw}(\mathcal{I})^{\mathrm{op}}} \mathrm{Fun}\left(|\mathcal{I}^\dagger_{/i}|, F(j)\right),$$
$$\operatorname*{laxcolim}_{\mathcal{I}^\dagger}{}^{\dagger} F \simeq \operatorname*{colim}_{(i\to j)\in \mathrm{Tw}(\mathcal{I})} |(\mathcal{I}^{\mathrm{op}})^\dagger_{/j}| \times F(i).$$

Here we use the twisted arrow category with projection $\mathrm{Tw}(\mathcal{I}) \to \mathcal{I} \times \mathcal{I}^{\mathrm{op}}$, $(i \to j) \mapsto (i, j)$. The formulas for the op-versions are obtained by replacing the localized slice categories by their opposites.

We mark $\mathrm{Un}^{\mathrm{co}} F \to \mathcal{I}$ and $\mathrm{Un}^{\mathrm{ct}} F \to \mathcal{I}^{\mathrm{op}}$ by the coCartesian resp. Cartesian lifts of marked edges. By [Ber24, Theorem 4.4],

$$\operatorname*{laxcolim}^{\dagger}_{\mathcal{I}^\dagger} F \simeq \left|\mathrm{Un}^{\mathrm{ct}} F\right| \qquad \text{and} \qquad \operatorname*{oplaxcolim}^{\dagger}_{\mathcal{I}^\dagger} F \simeq \left|\mathrm{Un}^{\mathrm{co}} F\right|$$
$$\operatorname*{laxlim}^{\dagger}_{\mathcal{I}^\dagger} F \simeq \mathrm{Fun}^{\dagger}_{/\mathcal{I}^\dagger}\left(\mathcal{I}^\dagger, \mathrm{Un}^{\mathrm{co}} F\right) \qquad \text{and} \qquad \operatorname*{oplaxlim}^{\dagger}_{\mathcal{I}^\dagger} F \simeq \mathrm{Fun}^{\dagger}_{/\mathcal{I}^{\dagger\,\mathrm{op}}}\left(\mathcal{I}^{\dagger\,\mathrm{op}}, \mathrm{Un}^{\mathrm{ct}} F\right).$$

In particular, the partially lax limit consists of those sections that send marked edges to coCartesian edges, also explaining Example 3.3.

Example 3.5. Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}})$ be an admissibility structure. Then, $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}})$ is a marked ∞-category. Our Theorem A concerns a partially lax limit over its opposite.

Remark 3.6 ([GLP26, Proposition 2.9])**.** Let $\mathrm{CMon}(\mathrm{Cat}_\infty)$ denote the ∞-category of symmetric monoidal ∞-categories and symmetric monoidal functors. Given a functor $F\colon \mathcal{I}^\dagger \to \mathrm{CMon}(\mathrm{Cat}_\infty)$ its partially lax limit $\operatorname{laxlim}^{\dagger}_{\mathcal{I}^\dagger} F$ admits a preferred symmetric monoidal structure, computed pointwise. It is characterized by the natural equivalence

$$\mathrm{Fun}^{\otimes\text{-lax}}\left(\mathcal{C}, \operatorname*{laxlim}^{\dagger}_{\mathcal{I}^\dagger} F\right) \simeq \operatorname*{laxlim}^{\dagger}_{\mathcal{I}^\dagger} \mathrm{Fun}^{\otimes\text{-lax}}(\mathcal{C}, F(-))$$

for every symmetric monoidal ∞-category $\mathcal{C}$, where $\mathrm{Fun}^{\otimes\text{-lax}}$ denotes the ∞-category of lax symmetric monoidal functors and monoidal natural transformations. Thus, a lax symmetric monoidal functor into the partially lax limit is the same datum as a partially lax cone of lax symmetric monoidal functors. It is symmetric monoidal if and only if all its components are symmetric monoidal. The same statements hold for (partially) oplax limits.

We end this recollection with the following symmetric monoidal version of [GLP26, Proposition 2.21].

Lemma 3.7. *Let $G\colon \mathcal{I} \to \mathrm{Cat}_\infty$ be a diagram of right adjoints and let $F\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be its associated diagram of left adjoints. Then, there exists an equivalence*

$$\operatorname*{oplaxlim}_{\mathcal{I}} G \simeq \operatorname*{laxlim}_{\mathcal{I}^{\mathrm{op}}} F.$$

If, moreover, G is a diagram of symmetric monoidal right adjoints and the induced oplax symmetric monoidal structures on the left adjoints are strong, then this equivalence is symmetric monoidal for the preferred symmetric monoidal structures.

Proof. As in [GLP26, Proposition 2.21], taking mates gives an equivalence of ∞-categories

$$\operatorname*{oplaxlim}_{\mathcal{I}} G = \mathrm{Nat}^{\mathrm{oplax}}(\Delta(*), G) \simeq \mathrm{Nat}^{\mathrm{lax}}(\Delta(*), F) = \operatorname*{laxlim}_{\mathcal{I}^{\mathrm{op}}} F.$$

For the symmetric monoidal statement, the adjunctions lift to symmetric monoidal categories and lax symmetric monoidal functors, see [HHLN23, Corollary C]. For every symmetric monoidal ∞-category $\mathcal{C}$, postcomposition therefore gives adjoint diagrams $\mathrm{Fun}^{\otimes\text{-lax}}(\mathcal{C}, F(-))$ and $\mathrm{Fun}^{\otimes\text{-lax}}(\mathcal{C}, G(-))$. Applying the first statement to these diagrams and using Remark 3.6 gives a natural equivalence

$$\mathrm{Fun}^{\otimes\text{-lax}}\left(\mathcal{C}, \operatorname*{oplaxlim}_{\mathcal{I}} G\right) \simeq \mathrm{Fun}^{\otimes\text{-lax}}\left(\mathcal{C}, \operatorname*{laxlim}_{\mathcal{I}^{\mathrm{op}}} F\right).$$

The claim follows by Yoneda. □

3.2. **Functors that send partially lax limits to limits.** Throughout this subsection, we denote by $\mathbb{C}$ either the ∞-category of presentably symmetric monoidal ∞-categories $\mathrm{CAlg}(\mathrm{Pr}^L)$ or its full subcategory $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ spanned by stable ∞-categories. We consider functors $\mathcal{R}\colon \mathbb{C} \to \mathcal{D}$, where $\mathcal{D}$ is any ∞-category. We think of $\mathcal{R}$ as extracting an invariant of presentably symmetric monoidal categories, e.g. the Picard group.

For a small marked ∞-category $\mathcal{I}^\dagger = (\mathcal{I}, \mathcal{W})$ and a functor $F\colon \mathcal{I} \to \mathbb{C}$, we equip $\operatorname{laxlim}^{\dagger}_{\mathcal{I}^\dagger} F$ with the preferred symmetric monoidal structure from Remark 3.6. The partially lax limit is presentable, with colimits computed pointwise, and its tensor product preserves colimits in each variable. It is stable if the categories $F(i)$ are stable. We therefore regard $\operatorname{laxlim}^{\dagger}_{\mathcal{I}^\dagger} F$ as an object of $\mathbb{C}$. The inclusion $\lim_{\mathcal{I}} F \to \operatorname{laxlim}^{\dagger}_{\mathcal{I}^\dagger} F$ from Example 3.3 is symmetric monoidal and preserves colimits, so it too is a morphism in $\mathbb{C}$.

Definition 3.8. We call $\mathcal{R}$ *lax-invariant* if, for every small marked ∞-category $\mathcal{I}^\dagger = (\mathcal{I}, \mathcal{W})$ and every functor $F\colon \mathcal{I} \to \mathbb{C}$, the map

$$\mathcal{R}\left(\lim_{\mathcal{I}} F\right) \longrightarrow \mathcal{R}\left(\operatorname*{laxlim}_{\mathcal{I}^\dagger} F\right) \tag{2}$$

induced by Example 3.3 is an equivalence.

Proposition 3.9. *Suppose that $\mathcal{R}$ preserves small limits. Then $\mathcal{R}$ is lax-invariant if and only if, for every $\mathcal{C} \in \mathbb{C}$, the constant-functor map induces an equivalence*

$$\mathcal{R}(\mathcal{C}) \longrightarrow \mathcal{R}\left(\mathrm{Fun}(\Delta^1, \mathcal{C})\right),$$

where the functor category is equipped with its pointwise symmetric monoidal structure.

Proof. Necessity follows from

$$\mathcal{C} \simeq \lim_{\Delta^1} \Delta(\mathcal{C}) \qquad \text{and} \qquad \mathrm{Fun}(\Delta^1, \mathcal{C}) \simeq \operatorname*{laxlim}_{\Delta^1} \Delta(\mathcal{C}).$$

Conversely, suppose that the displayed map is an equivalence for every $\mathcal{C}$. For a small ∞-category $\mathcal{K}$, let $\mathcal{K}^{\mathrm{gpd}}$ denote its groupoid completion. Restriction induces a natural map

$$\mathcal{R}\left(\mathrm{Fun}(\mathcal{K}^{\mathrm{gpd}}, \mathcal{C})\right) \longrightarrow \mathcal{R}\left(\mathrm{Fun}(\mathcal{K}, \mathcal{C})\right).$$

Both sides send colimits in $\mathcal{K}$ to limits: groupoid completion preserves colimits, functor categories turn colimits in their first variable into limits in $\mathbb{C}$, and $\mathcal{R}$ preserves limits. The map is an equivalence for $\mathcal{K} = \Delta^1$ by assumption. Consequently, it is an equivalence for every $\mathcal{K}$, since Cat_∞ is generated under small colimits by Δ^1. Indeed, Cat_∞ is generated by the simplices Δ^n by [Lur18, Example 20.4.1.9]; moreover, Δ^0 is a retract of Δ^1, and for $n \geq 1$ we have $\Delta^n \simeq \Delta^1 \amalg_{\Delta^0} \cdots \amalg_{\Delta^0} \Delta^1$ in Cat_∞.

In particular, if $\mathcal{K}^{\mathrm{gpd}}$ is contractible, the constant-functor map

$$\mathcal{R}(\mathcal{C}) \longrightarrow \mathcal{R}\left(\mathrm{Fun}(\mathcal{K}^{\mathrm{gpd}}, \mathcal{C})\right) \longrightarrow \mathcal{R}\left(\mathrm{Fun}(\mathcal{K}, \mathcal{C})\right)$$

is an equivalence.

This applies to $\mathcal{K} = |\mathcal{I}^\dagger_{/i}|$: localization does not change groupoid completion, and $\mathcal{I}_{/i}$ has a terminal object. The end formula from Remark 3.4, with pointwise symmetric monoidal structures, therefore gives

$$\begin{aligned}\mathcal{R}\left(\operatorname*{laxlim}_{\mathcal{I}^\dagger} F\right) &\simeq \lim_{(i\to j)\in \mathrm{Tw}(\mathcal{I})^{\mathrm{op}}} \mathcal{R}\left(\mathrm{Fun}\left(|\mathcal{I}^\dagger_{/i}|, F(j)\right)\right) \\ &\simeq \lim_{(i\to j)\in \mathrm{Tw}(\mathcal{I})^{\mathrm{op}}} \mathcal{R}(F(j)) \simeq \lim_{\mathcal{I}} (\mathcal{R}\circ F) \simeq \mathcal{R}\left(\lim_{\mathcal{I}} F\right).\end{aligned}$$

Here the third equivalence uses that the target projection $\mathrm{Tw}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ is limit-cofinal. These equivalences identify (2) with an equivalence. □

Proposition 3.10. *The functor*

$$(-)^{\mathrm{dbl}}\colon \mathrm{CAlg}(\mathrm{Pr}^L) \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

taking the full subcategory of dualizable objects preserves small limits and is lax-invariant. In particular, there is a preferred symmetric monoidal equivalence

$$\lim_{i\in\mathcal{I}} F(i)^{\mathrm{dbl}} \xrightarrow{\simeq} \left(\operatorname*{laxlim}_{\mathcal{I}^\dagger} F\right)^{\mathrm{dbl}}.$$

Proof. In a limit of symmetric monoidal ∞-categories, dualizability is detected componentwise. Thus, $(-)^{\mathrm{dbl}}$ preserves small limits.

By Proposition 3.9, it remains to show that

$$\mathcal{C}^{\mathrm{dbl}} \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{dbl}}$$

is an equivalence. We claim that an arrow $f\colon X \to Y$ is dualizable for the pointwise tensor product precisely when X, Y are dualizable and f is an equivalence.

Suppose that f is dualizable. Evaluation at the endpoints shows that X and Y are dualizable, and we may write its dual as $g\colon \mathbb{D}(X) \to \mathbb{D}(Y)$. Let

$$\mathrm{ev}_X\colon \mathbb{D}(X) \otimes X \to \mathbf{1}, \qquad \mathrm{coev}_X\colon \mathbf{1} \to X \otimes \mathbb{D}(X)$$

denote the duality maps, and similarly for Y. Naturality of evaluation and coevaluation in the arrow category gives

$$\mathrm{ev}_Y \circ (g \otimes f) \simeq \mathrm{ev}_X, \qquad (f \otimes g) \circ \mathrm{coev}_X \simeq \mathrm{coev}_Y.$$

We claim that g and $\mathbb{D}(f)\colon \mathbb{D}(Y) \to \mathbb{D}(X)$ are mutually inverse. By the defining identities for $\mathbb{D}(f)$,

$$\begin{aligned}\mathrm{ev}_X \circ \big((\mathbb{D}(f) \circ g) \otimes \mathrm{id}_X\big) &\simeq \mathrm{ev}_Y \circ (g \otimes f) \simeq \mathrm{ev}_X,\\ \big(\mathrm{id}_Y \otimes (g \circ \mathbb{D}(f))\big) \circ \mathrm{coev}_Y &\simeq (f \otimes g) \circ \mathrm{coev}_X \simeq \mathrm{coev}_Y.\end{aligned}$$

The triangle identities imply that evaluation and coevaluation induce equivalences

$$\begin{aligned}\mathrm{Map}(\mathbb{D}(X), \mathbb{D}(X)) &\simeq \mathrm{Map}(\mathbb{D}(X) \otimes X, \mathbf{1}),\\ \mathrm{Map}(\mathbb{D}(Y), \mathbb{D}(Y)) &\simeq \mathrm{Map}(\mathbf{1}, Y \otimes \mathbb{D}(Y)).\end{aligned}$$

Hence $\mathbb{D}(f) \circ g \simeq \mathrm{id}_{\mathbb{D}(X)}$ and $g \circ \mathbb{D}(f) \simeq \mathrm{id}_{\mathbb{D}(Y)}$. Thus $\mathbb{D}(f)$, and therefore f, is an equivalence.

Conversely, an equivalence $f\colon X \to Y$ between dualizable objects is equivalent to id_X in $\mathrm{Fun}(\Delta^1, \mathcal{C})$, which is dualizable since X is. □

Corollary 3.11. *The Picard spectrum, the Picard space and the Picard group functors are lax-invariant.*

Proof. These functors factor through the lax-invariant functor $(-)^{\mathrm{dbl}}$ from Proposition 3.10. □

Proposition 3.12. *Let* $\mathrm{CAlg}^{\mathrm{rig}} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ *denote the full subcategory of rigid* ∞*-categories in the sense of Gaitsgory–Rozenblyum* [GR17, HSSS21, Ram26]*. Rigidification*

$$\mathrm{Rig}\colon \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}) \longrightarrow \mathrm{CAlg}^{\mathrm{rig}}$$

preserves small limits and is lax-invariant. Thus, for every small marked ∞*-category* $\mathcal{I}^\dagger$ *and every* $F\colon \mathcal{I} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$*, the canonical map*

$$\mathrm{Rig}\left(\lim_{\mathcal{I}} F\right) \longrightarrow \mathrm{Rig}\left(\operatorname*{laxlim}_{\mathcal{I}^\dagger}{}^{\dagger} F\right)$$

is an equivalence.

Proof. Rigidification is right adjoint to the inclusion of rigid categories [Ram26, Theorem 4.81], and hence preserves limits with values in $\mathrm{CAlg}^{\mathrm{rig}}$. By Proposition 3.9, it suffices to verify the constant-arrow condition.

Let $\mathcal{A}$ be rigid and consider the map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})}(\mathcal{A}, \mathcal{C}) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})}\left(\mathcal{A}, \mathrm{Fun}(\Delta^1, \mathcal{C})\right)$$

induced by the constant-functor map. A symmetric monoidal left adjoint $\mathcal{A} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is equivalently a monoidal natural transformation between two symmetric monoidal left adjoints $\mathcal{A} \to \mathcal{C}$. Such a transformation is an equivalence by [Ram26, Lemma 4.94]. Thus, the map on mapping spaces is an equivalence, so the universal property of rigidification and Yoneda give

$$\mathrm{Rig}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Rig}\left(\mathrm{Fun}(\Delta^1, \mathcal{C})\right),$$

as required. This is the same argument underlying the lax pullback result [Ram26, Corollary 4.95]. □

Remark 3.13. The target in Proposition 3.12 matters: the limit of rigidifications is taken in $\mathrm{CAlg}^{\mathrm{rig}}$. The forgetful functor to $\widehat{\mathrm{Cat}}_\infty$ does not generally preserve these limits, see [Ram26, Example 4.88].

4. Presheaf Categories as Partially Lax Limits

The goal of this section is to identify presheaf categories of admissibility structures with a partially lax limit, i.e. to prove Theorem B.

The Lax Limit. For this section we consider a small ∞-category $\mathcal{G}$ together with an admissibility structure $\mathcal{G}^{\mathrm{ad}}$ in the sense of Definition 2.1. Moreover, fix a cocompletely symmetric monoidal ∞-category $\mathcal{C}$, i.e. it has all colimits and the tensor product commutes with colimits in each variable.

Since the coCartesian unstraightening of $\mathcal{G}^{\mathrm{op}} \to \mathrm{Cat}_\infty,\ X \mapsto (\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}}$ is the opposite of the target functor $t\colon \mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}) \to \mathcal{G}$, the oplax colimit of $X \mapsto (\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}}$ is

$$\operatorname*{oplaxcolim}_{X\in\mathcal{G}^{\mathrm{op}}} (\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})^{\mathrm{op}},$$

by Remark 3.4; with initial oplax cocone

$$(\mathcal{G}^{\mathrm{ad}}_{/-})^{\mathrm{op}} \xRightarrow{\text{oplax}} \mathrm{const}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})^{\mathrm{op}}), \qquad \left\{\mathrm{fgt}^{\mathrm{op}}_X\colon (\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}} \to \mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})^{\mathrm{op}}\right\}_{X\in\mathcal{G}^{\mathrm{op}}}.$$

For our fixed cocompletely symmetric monoidal ∞-category $\mathcal{C}$ there is an $(\infty,2)$-functor

$$\mathrm{Fun}(-,\mathcal{C})\colon \mathbb{C}\mathrm{at}^{\text{1-op}} \longrightarrow \mathbb{C}\mathrm{at}^{\otimes}_R, \qquad \mathcal{D} \mapsto \mathrm{Fun}(\mathcal{D},\mathcal{C}) \tag{3}$$

which equips functor categories with the pointwise symmetric monoidal structure and precomposition functoriality. Here $\mathbb{C}\mathrm{at}^{\otimes}_R$ is the $(\infty,2)$-category of large symmetric monoidal ∞-categories and symmetric monoidal right adjoint functors, and $(-)^{\text{1-op}}$ means that the 1-morphisms are reversed. For these constructions and the language of $(\infty,2)$-categories we refer to [BB25, Section 2 and Appendix A].

The functor from Equation (3) sends oplax colimits to oplax limits by the 2-op version of [LNP25, Proposition 4.11]. Thus, we have constructed a symmetric monoidal equivalence:

$$\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})) \xrightarrow{\simeq} \operatorname*{oplaxlim}_{X\in\mathcal{G}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}), \quad \Phi \mapsto \left((\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}} \xrightarrow{\mathrm{fgt}} \mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}\right)_{X\in\mathcal{G}}, \tag{4}$$

where $\mathcal{P}^{\mathcal{C}}\colon \mathcal{D} \mapsto \mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{C})$ denotes $\mathcal{C}$-valued presheaves.

Lemma 4.1. *Let $f\colon X \to Y$ be a morphism in $\mathcal{G}$. Recall that the left adjoint $\mathrm{Lan}_{(f^*)^{\mathrm{op}}}$ of*

$$\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}) \longrightarrow \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/Y}), \qquad \Phi \mapsto \left((\mathcal{G}^{\mathrm{ad}}_{/Y})^{\mathrm{op}} \xrightarrow{(f^*)^{\mathrm{op}}} (\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}\right)$$

inherits an oplax symmetric monoidal structure. Then, $\mathrm{Lan}_{(f^*)^{\mathrm{op}}}$ is strong symmetric monoidal.

Proof. Note that $(f^*)^{\mathrm{op}}\colon (\mathcal{G}^{\mathrm{ad}}_{/Y})^{\mathrm{op}} \to (\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}}$ is symmetric monoidal for the coCartesian symmetric monoidal structures. By [Aok23, Proposition 2.21] the pointwise symmetric monoidal structure on $\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$ agrees with Day convolution with the cartesian monoidal structure on $\mathcal{G}^{\mathrm{ad}}_{/X}$. Viewed via Day convolution, the oplax symmetric monoidal structure on

$$\mathrm{Lan}_{(f^*)^{\mathrm{op}}}\colon \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/Y}) \longrightarrow \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$$

is symmetric monoidal by [Nik16, Corollary 3.8] and [Chi21, Proposition 6.18 and its proof]. □

Therefore, we can apply the mate equivalence (Lemma 3.7) to our functor $\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})\colon \mathcal{G} \to \mathbb{C}\mathrm{at}$ to obtain a diagram

$$\mathcal{G}^{\mathrm{op}} \longrightarrow \mathbb{C}\mathrm{at}^{\otimes}, \qquad X \mapsto \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}) \tag{5}$$

with transition maps given by componentwise left adjoints, i.e. $\mathrm{Lan}_{(f^*)^{\mathrm{op}}}$, and an equivalence

$$\operatorname*{oplaxlim}_{X\in\mathcal{G}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}) \xrightarrow{\simeq} \operatorname*{laxlim}_{X\in\mathcal{G}^{\mathrm{op}}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$$

of symmetric monoidal ∞-categories. Putting this together with Equation (4) we arrive at:

Lemma 4.2. *There is a symmetric monoidal equivalence*

$$\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})) \xrightarrow{\simeq} \operatorname*{laxlim}_{X\in\mathcal{G}^{\mathrm{op}}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}), \quad \Phi \mapsto \left((\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}} \xrightarrow{\mathrm{fgt}} \mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}\right)_{X\in\mathcal{G}^{\mathrm{op}}}. \tag{6}$$

The Partially Lax Subcategory. By Remark 3.6, the partially lax limit of the functor $\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ in Equation (5) is the symmetric monoidal full subcategory of $\operatorname{laxlim}_{X\in\mathcal{G}^{\mathrm{op}}}\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$ consisting of those objects of the lax limit such that the structure maps at admissible morphisms are equivalences.

Observation 4.3. *Since admissible morphisms are closed under composition, for any admissible morphism $f\colon X\to Y$ the functor*

$$f^*\colon \mathcal{G}^{\mathrm{ad}}_{/Y}\longrightarrow\mathcal{G}^{\mathrm{ad}}_{/X}$$

is right adjoint to $f\circ(-)\colon \mathcal{G}^{\mathrm{ad}}_{/X}\to\mathcal{G}^{\mathrm{ad}}_{/Y}$. Consequently, $\operatorname{Lan}_{(f^*)^{\mathrm{op}}}$ identifies with

$$\mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/Y}\right)\longrightarrow\mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right),\qquad \Phi\mapsto\left((\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}}\xrightarrow{f\circ(-)}(\mathcal{G}^{\mathrm{ad}}_{/Y})^{\mathrm{op}}\xrightarrow{\Phi}\mathcal{C}\right).$$

We apply the observation to describe the full subcategory defining the partially lax limit: Under the equivalence (6), the partially lax limit is equivalent to the symmetric monoidal full subcategory of $\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))$ spanned by those $\Phi\in\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))$ such that for every admissible $a\colon A\to Y$ and admissible $f\colon Y\to X$ the natural map

$$\Phi(f\circ a)\longrightarrow\Phi(a)$$

is an equivalence, where we use Observation 4.3 to identify the source with $\Phi(f\circ a)$. This map is Φ applied to the morphism $a\to(f\circ a)$ in $\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})$ given by

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ Y & & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f} \\ & & X \end{array}$$

with the constant homotopy filling the diagram. If Φ satisfies this condition, we say that Φ is *partially lax*.

Precomposition by the source functor $s\colon \mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})\to\mathcal{G}$ induces a symmetric monoidal functor

$$(s^{\mathrm{op}})^*\colon \mathcal{P}^{\mathcal{C}}(\mathcal{G})\longrightarrow\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))$$

for the pointwise symmetric monoidal structures. This allows us to state a precise version of Theorem B.

Theorem 4.4. *Let $\mathcal{G}$ be a small ∞-category and $\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})\subseteq\mathrm{Ar}(\mathcal{G})$ an admissible class of morphisms. Let $\mathcal{C}$ be a cocompletely symmetric monoidal ∞-category. The symmetric monoidal composite functor*

$$\mathcal{P}^{\mathcal{C}}(\mathcal{G})\xrightarrow{(s^{\mathrm{op}})^*}\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))\xrightarrow[(6)]{\simeq}\operatorname*{laxlim}_{X\in\mathcal{G}^{\mathrm{op}}}\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X}),\quad \Phi\mapsto\left((\mathcal{G}^{\mathrm{ad}}_{/X})^{\mathrm{op}}\xrightarrow{\mathrm{fgt}}\mathcal{G}^{\mathrm{op}}\xrightarrow{\Phi}\mathcal{C}\right)_{X\in\mathcal{G}^{\mathrm{op}}}$$

is fully faithful and has essential image the symmetric monoidal full subcategory $\operatorname{laxlim}^{\dagger}{}_{X\in\mathcal{G}^{\mathrm{op}}}\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$.

Proof. Consider the functors

$$\mathcal{G}\xrightarrow{\mathrm{const}}\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G})\xrightarrow{s}\mathcal{G}$$

with const $\dashv s$ and adjunction counit evaluated at an admissible morphism $a:A\to B$ depicted by the morphism $\mathrm{id}_A\to a$ in the arrow category:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ \downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow[a]{} & B \end{array}$$

It induces functors

$$\mathcal{P}^{\mathcal{C}}(\mathcal{G})\xrightarrow{s^*}\mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))\xrightarrow{\mathrm{const}^*}\mathcal{P}^{\mathcal{C}}(\mathcal{G})$$

so that in particular s^* is fully faithful. It suffices to show that the essential image of s^* consists precisely of the partially lax presheaves.

First, let $F \in \mathcal{P}^{\mathcal{C}}(\mathcal{G})$. Then, $s^*F \in \mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))$ lies in the partially lax limit. Indeed, let $f\colon B \to C$ be an admissible morphism. We need to check that the map $F(A) = s^*F(f \circ a) \to s^*F(a) = F(A)$ is an equivalence, but it is the identity. Conversely, we need to check that any partially lax $\Phi \in \mathcal{P}^{\mathcal{C}}(\mathrm{Ar}^{\mathrm{ad}}(\mathcal{G}))$ is in the image of s^*. Indeed, the counit of const $\dashv s$ induces a natural transformation $\varepsilon^*\colon \Phi \Rightarrow s^* \operatorname{const}^* \Phi$ which pointwise evaluates to $\Phi(a) = \Phi(a \circ \mathrm{id}_{s(a)}) \to \Phi(\mathrm{id}_{s(a)})$. This is an equivalence because Φ is partially lax. So $\Phi \simeq s^* \operatorname{const}^* \Phi$. □

5. Sheaf Categories as Partially Lax Limits

We will now pass from presheaves to sheaves, and prove Theorem A. Fix a geometric site $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$.

Lemma 5.1. *Let $\mathcal{C}$ be a complete ∞-category. A presheaf $\Phi\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ is a sheaf if and only if for all $X \in \mathcal{G}$ the composite functor*

$$\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right)^{\mathrm{op}} \hookrightarrow \left(\mathcal{G}_{/X}\right)^{\mathrm{op}} \longrightarrow \mathcal{G}^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}$$

is a sheaf.

Proof. In the case of $\mathcal{C} = \mathrm{An}$ this is [Lur18, Prop. 20.6.3.1].[6] Lurie's proof goes through for a general target category $\mathcal{C}$ as it only uses cofinality arguments. □

We can sheafify the pointwise monoidal structures on presheaf categories to monoidal structures on sheaf categories. With this structure we now prove our Theorem A.

Theorem 5.2. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. There is a functor*

$$\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/-}; \mathcal{C})\colon \mathcal{G}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}^{\otimes}, \qquad X \mapsto \mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/X}; \mathcal{C})$$

to the ∞-category of large symmetric monoidal ∞-categories and a natural transformation

$$L\colon \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-}) \Longrightarrow \mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/-}; \mathcal{C}) \qquad \text{of functors } \mathcal{G}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}^{\otimes}$$

whose components are τ-sheafification. Moreover, the induced symmetric monoidal functor

$$\operatorname{laxlim}^{\dagger} L\colon \mathcal{P}^{\mathcal{C}}(\mathcal{G}) \longrightarrow \operatorname*{laxlim}^{\dagger}_{X \in \mathcal{G}^{\mathrm{op}}} \mathrm{Sh}\left(\mathcal{G}^{\mathrm{ad}}_{/X}; \mathcal{C}\right)$$

admits a fully faithful right adjoint with essential image the τ-sheaves.

Proof. We closely follow the argument of [LNP25, Lemma 4.12] in this proof. Let us first ignore the symmetric monoidal structures. Consider the coCartesian unstraightening $\mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-}) \to \mathcal{G}^{\mathrm{op}}$ of the functor $\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})\colon \mathcal{G}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, $X \mapsto \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$ that we are taking a partially lax limit over in Theorem 4.4. Let $\mathcal{E} \subseteq \mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ denote the full subcategory spanned by $\mathrm{Sh}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})$ for all $X \in \mathcal{G}$.

First, the composite $\mathcal{E} \to \mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-}) \to \mathcal{G}^{\mathrm{op}}$ is a coCartesian fibration, as shown in the proof of [LNP25, Lemma 4.12].[7] This uses that for $f\colon X \to Y$ the induced map

$$\mathrm{Lan}_{(f^*)^{\mathrm{op}}}\colon \mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/Y}\right) \longrightarrow \mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right)$$

preserves L-equivalences, where L denotes sheafification. By an adjunction argument this is equivalent to checking that the associated right adjoint f^* preserves sheaves. This follows because the pullback of a τ-cover is again a τ-cover. The straightening of $\mathcal{E} \to \mathcal{G}^{\mathrm{op}}$ is the desired functor $\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/-}; \mathcal{C})$.

Now, the map $\mathcal{E} \to \mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ fiberwise has a left adjoint. We wish to show that it assembles into a relative left adjoint $\widetilde{L}\colon \mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-}) \to \mathcal{E}$. By [BH21, Proposition D.7] it suffices to check for $f\colon X \to Y$ the induced map $\mathrm{Lan}_{(f^*)^{\mathrm{op}}}\colon \mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/Y}\right) \longrightarrow \mathcal{P}^{\mathcal{C}}\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right)$ preserves L-equivalences, which we just checked. Moreover, $\widetilde{L}$ is automatically a map of coCartesian fibrations, see e.g. the proof of [Hil24, Proposition 2.2.5(2)]. By straightening $\widetilde{L}$ we have thus built the natural transformation L claimed in the theorem.

[6] To translate his result observe that a presheaf is a sheaf if it satisfies descent for every cover of every $X \in \mathcal{G}$ and this can be checked in the slice category over X.

[7] But note that $\mathcal{E} \to \mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ does not need to preserve all coCartesian edges. More precisely, [LNP25, Lemma 4.12] shows that a coCartesian lift is given by a coCartesian lift in $\mathrm{Un}^{\mathrm{co}} \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ followed by the localization map in the fiber.

Furthermore, the inclusion $\mathcal{E} \hookrightarrow \mathrm{Un}^{\mathrm{co}}\, \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ preserves coCartesian lifts of admissibles. For this, we need to show that for every admissible morphism $f\colon X \to Y$ the functor $\mathrm{Lan}_{(f^*)^{\mathrm{op}}}$ preserves τ-sheaves. By Observation 4.3 this amounts to showing that if $\Phi\colon (\mathcal{G}^{\mathrm{ad}}_{/Y})^{\mathrm{op}} \to \mathcal{C}$ is a τ-sheaf, then the composite

$$\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right)^{\mathrm{op}} \xrightarrow{f\circ -} \left(\mathcal{G}^{\mathrm{ad}}_{/Y}\right)^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}$$

is a τ-sheaf as well. This holds as τ-covers in $\mathcal{G}^{\mathrm{ad}}_{/Z}$ are detected in $\mathcal{G}$ for all $Z \in \mathcal{G}$.

By Remark 3.4 the partially lax limits of $\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/-}; \mathcal{C})$ and $\mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ are computed by the ∞-categories of those sections of the respective cocartesian fibrations that send admissible morphisms to coCartesian morphisms and $\mathrm{laxlim}^{\dagger} L$ is given by postcomposition with the coCartesian edges preserving functor $\tilde{L}$. Therefore, postcomposition with the inclusion $\mathcal{E} \hookrightarrow \mathrm{Un}^{\mathrm{co}}\, \mathcal{P}(\mathcal{G}^{\mathrm{ad}}_{/-})$ gives a fully faithful right adjoint to $\mathrm{laxlim}^{\dagger} L$.

So the essential image of the right adjoint to $\mathrm{laxlim}^{\dagger} L$ consists of those presheaves $\Phi\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ satisfying for all $X \in \mathcal{G}$ that the composite functor

$$\left(\mathcal{G}^{\mathrm{ad}}_{/X}\right)^{\mathrm{op}} \hookrightarrow (\mathcal{G}_{/X})^{\mathrm{op}} \longrightarrow \mathcal{G}^{\mathrm{op}} \xrightarrow{\Phi} \mathcal{C}$$

is a sheaf. Lemma 5.1 implies that the essential image of the right adjoint consists precisely of the τ-sheaves.

Finally, we apply [LNP25, Corollary 4.14] to $F \coloneqq \mathcal{P}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/-})$ and $\left\{GX \coloneqq \mathrm{Sh}^{\mathcal{C}}(\mathcal{G}^{\mathrm{ad}}_{/X})\right\}_{X\in\mathcal{G}}$ to lift L to a symmetric monoidal transformation. □

Remark 5.3. The analogous statement of Theorem 5.2 holds for sheaves replaced by hypersheaves. Indeed, the proof of Theorem 5.2 goes through in this setting.

Remark 5.4. By [LNP25, Theorem 5.10], we can also pass to modules over any $R \in \mathrm{CAlg}(\mathrm{Sh}(\mathcal{G}; \mathcal{C}))$ in Theorem 5.2 to obtain a symmetric monoidal equivalence

$$\mathrm{Mod}_R\left(\mathrm{Sh}(\mathcal{G}; \mathcal{C})\right) \xrightarrow{\simeq} \operatorname*{laxlim}^{\dagger}_{X\in\mathcal{G}^{\mathrm{op}}} \mathrm{Mod}_{R_X}\left(\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/X}; \mathcal{C})\right),$$

where R_X denotes the restriction of R to $\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/X}; \mathcal{C})$.

Corollary 5.5. *Let $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ be a geometric site with terminal object $* \in \mathcal{G}$ and $\mathcal{R}\colon \mathbb{C} \to \mathcal{D}$ be a lax-invariant functor, where $\mathbb{C} \in \{\mathrm{CAlg}(\mathrm{Pr}^L), \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})\}$ and $\mathcal{D}$ is any ∞-category. Let $\mathcal{C} \in \mathbb{C}$. Then, the restriction functor induces an equivalence*

$$\mathcal{R}(\mathrm{Sh}(\mathcal{G}; \mathcal{C})) \xrightarrow{\simeq} \mathcal{R}\left(\mathrm{Sh}(\mathcal{G}^{\mathrm{ad}}_{/*}; \mathcal{C})\right).$$

Proof. Apply Theorem A. The limit is then computed on the initial object $* \in \mathcal{G}^{\mathrm{op}}$. □

We now apply Theorem A to our condensed example (Example 2.5).

Corollary 5.6. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Then, there is an equivalence*

$$\mathrm{Cond}(\mathcal{C}) \simeq \operatorname*{laxlim}^{\dagger}_{X\in\mathrm{ExtrDisc}^{\mathrm{op}}} \mathrm{Sh}(X; \mathcal{C})$$

of symmetric monoidal ∞-categories, where the open embeddings are marked. In particular, there is an equivalence $\mathcal{R}(\mathrm{Cond}(\mathcal{C})) \simeq \mathcal{R}(\mathcal{C})$ for lax-invariant functors $\mathcal{R}$.

Proof. A geometric site structure on ExtrDisc was exhibited in [RZ26, Theorem A] and moreover the associated slice categories were computed in [RZ26, Theorem B]. Combined with Theorem A we arrive at the desired. The last part follows from Corollary 5.5. □

We obtain similar descriptions in the setting of Example 2.3 and Example 2.4.

6. Gros étale exodromy

We recall that the *condensed Galois ∞-category* of X from [BGH20, HHL+25, Mai26, vDdB26] is

$$\mathrm{Gal}(X)\colon \mathrm{ExtrDisc}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty,\ S \mapsto \mathrm{Fun}^*_{\mathrm{loc\,coh}}(\mathrm{Sh}(X_{\text{ét}}), \mathrm{Sh}(S)),$$

which for $S \in \mathrm{ExtrDisc}^{\mathrm{op}}$ consists of locally coherent geometric morphisms $\mathrm{Sh}(S) \to \mathrm{Sh}(X_{\text{ét}})$, see [vDdB26, 2.1.6]. It is functorial in X because maps of schemes induce locally coherent morphisms between their associated étale topoi. This object plays a central role in Barwick–Glasman–Haine's exodromy philosophy [BGH20]. Before stating it, we need to recall some notation about condensed ∞-categories.

(1) For two functors $\mathcal{C}, \mathcal{D}\colon \mathrm{ExtrDisc}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ we denote by

$$\mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D}) \coloneqq \mathrm{Nat}(\mathcal{C}, \mathcal{D}) \simeq \lim_{(S\to S')\in \mathrm{Tw}(\mathrm{ExtrDisc}^{\mathrm{op}})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}(S), \mathcal{D}(S'))$$

the ∞-category of *continuous functors* from $\mathcal{C}$ to $\mathcal{D}$, see [vDdB26, Definition 3.1.1].

(2) Let $\mathcal{C}$ be a compactly assembled presentable ∞-category. The condensed ∞-category $\underline{\mathcal{C}}$ is defined by

$$\underline{\mathcal{C}}\colon \mathrm{ExtrDisc}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty,\ S \mapsto \mathrm{Sh}(S; \mathcal{C}),$$

see [vDdB26, Definition 3.2.4] for more details.

Work of van Dobben de Bruyn provides a (petit version) of étale exodromy:

Fact 6.1 ([vDdB26, Theorem 2])**.** *Let X be a scheme, and let $\mathcal{C}$ be a compactly assembled presentable ∞-category. Then, the pro-étale exodromy correspondence restricts to an étale exodromy correspondence*

$$\mathrm{Ex}^{\acute{e}t}\colon \mathrm{Sh}^{\mathrm{post}}(X_{\acute{e}t}; \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \underline{\mathcal{C}}).$$

Here, the pro-étale exodromy map is constructed in the proof of [vDdB26, Theorem 1].

Remark 6.2. One can check by hand that $\mathrm{Ex}^{\text{ét}}$ is natural in X, by first checking a 1-categorical naturality and then applying functorial ∞-categorical machinery.

We will now use Theorem A to deduce a gros version of this result. Throughout, let us denote by Sch the (1-)category of κ-small qcqs schemes.

Definition 6.3. Let the *gros Galois ∞-category* be

$$\mathrm{GAL} \coloneqq \operatorname*{laxcolim}^{\dagger}_{X\in \mathrm{Sch}} \mathrm{Gal}(X) \in \mathrm{Fun}\left(\mathrm{ExtrDisc}^{\mathrm{op}}, \mathrm{Cat}_\infty\right),$$

where we mark the étale morphisms, and the partially lax colimit is taken pointwise.

Remark 6.4. For our purposes, it is not needed whether GAL is a condensed ∞-category, i.e. a sheaf, and we do not know whether this is true.

Theorem 6.5. *Let $\mathcal{C}$ be a compactly assembled presentable ∞-category. Then, there is a preferred étale exodromy equivalence*

$$\mathrm{Sh}^{\mathrm{post}}(\mathrm{Sch}_{\acute{e}t}; \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{GAL}, \underline{\mathcal{C}})$$

of ∞-categories.

Proof. It's enough to prove that the n-truncated version

$$\mathrm{Sh}(\mathrm{Sch}_{\text{ét}}; \mathrm{An}_{\leq n}) \longrightarrow \mathrm{Fun}^{\mathrm{cts}}(\mathrm{GAL}, \underline{\mathrm{An}}_{\leq n})$$

is an equivalence. As in [vDdB26, Theorem 3.2.6] we can pass to the limit and tensor by $\mathcal{C}$ to finish.

Applying $\mathrm{laxlim}^{\dagger}$ to Fact 6.1 combined with our Theorem A we deduce

$$\mathrm{Sh}(\mathrm{Sch}_{\text{ét}}; \mathrm{An}_{\leq n}) \xrightarrow{\simeq} \operatorname*{laxlim}^{\dagger}_{X\in \mathrm{Sch}^{\mathrm{op}}} \mathrm{Sh}(X_{\text{ét}}; \mathrm{An}_{\leq n}) \xrightarrow{\simeq} \operatorname*{laxlim}^{\dagger}_{X\in \mathrm{Sch}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \underline{\mathrm{An}}_{\leq n}),$$

so we are left to identify the right side. We do this here:

$$\begin{aligned}
\operatorname*{laxlim}^{\dagger}_{X\in\mathrm{Sch}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X),\underline{\mathrm{An}}_{\leq n}) &\simeq \lim_{(X\to X')\in \mathrm{Tw}(\mathrm{Sch}^{\mathrm{op}})^{\mathrm{op}}} \mathrm{Fun}\left(|\,\mathrm{Sch}^{\mathrm{op}\,\dagger}_{/X}\,|, \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X')(-),\underline{\mathrm{An}}_{\leq n})\right)\\
&\simeq \lim_{X\to X'}\lim_{(S\to S')\in \mathrm{Tw}(\mathrm{ExtrDisc}^{\mathrm{op}})^{\mathrm{op}}} \mathrm{Fun}\left(|\,\mathrm{Sch}^{\mathrm{op}\,\dagger}_{/X}\,|, \mathrm{Fun}(\mathrm{Gal}(X')(S),\underline{\mathrm{An}}_{\leq n}(S'))\right)\\
&\simeq \lim_{X\to X'}\lim_{S\to S'} \mathrm{Fun}\left(\mathrm{Gal}(X')(S)\times|\,\mathrm{Sch}^{\mathrm{op}\,\dagger}_{/X}\,|, \underline{\mathrm{An}}_{\leq n}(S')\right)\\
&\simeq \lim_{S\to S'}\lim_{X\to X'} \mathrm{Fun}\left(\mathrm{Gal}(X')(S)\times|\,\mathrm{Sch}^{\mathrm{op}\,\dagger}_{/X}\,|, \underline{\mathrm{An}}_{\leq n}(S')\right)\\
&\simeq \lim_{S\to S'} \mathrm{Fun}\left(\operatorname*{colim}_{(X\to X')\in \mathrm{Tw}(\mathrm{Sch}^{\mathrm{op}})} \mathrm{Gal}(X')(S)\times|\,\mathrm{Sch}^{\mathrm{op}\,\dagger}_{/X}\,|, \underline{\mathrm{An}}_{\leq n}(S')\right)\\
&\simeq \mathrm{Fun}^{\mathrm{cts}}\left(\operatorname*{laxcolim}^{\dagger}_{X} \mathrm{Gal}(X), \underline{\mathrm{An}}_{\leq n}\right),
\end{aligned}$$

as desired. □

Remark 6.6. The same proof goes through for Sch_T with κ-small schemes qcqs over an arbitrary base scheme T instead of Sch. So let $\mathrm{GAL}_T \coloneqq \operatorname{laxcolim}^{\dagger}{}_{X\in\mathrm{Sch}_T} \mathrm{Gal}(X)$, then there is an equivalence

$$\mathrm{Sh}^{\mathrm{post}}(\mathrm{Sch}_{T,\text{ét}};\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathrm{GAL}_T,\underline{\mathcal{C}})$$

of ∞-categories.

References

[Aok23] Ko Aoki. Tensor triangular geometry of filtered objects and sheaves. *Math. Z.*, 303(3):Paper No. 62, 27, 2023. (Cited on page 9.)

[BB25] Max Blans and Thomas Blom. On the chain rule in Goodwillie calculus, 2025. arXiv:2410.20504. (Cited on pages 3 and 9.)

[Ber24] John D. Berman. On lax limits in ∞-categories. *Proc. Amer. Math. Soc.*, 152(12):5055–5066, 2024. (Cited on pages 5 and 6.)

[BGH20] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy, 2020. arXiv:1807.03281. (Cited on pages 3 and 13.)

[BH19] Clark Barwick and Peter Haine. Pyknotic objects, I. Basic notions, 2019. arXiv:1904.09966. (Cited on pages 3 and 4.)

[BH21] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, (425):ix+207, 2021. (Cited on page 11.)

[Chi21] Michael Ching. ∞-operads and Day Convolution in Goodwillie Calculus. *Journal of the London Mathematical Society*, 104(3):1204–1249, 2021. (Cited on page 9.)

[Clo25] Adrian Clough. Differentiable sheaves I: fractured ∞-toposes and compactness. *Proc. Amer. Math. Soc.*, 153(10):4487–4500, 2025. (Cited on page 4.)

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus. Lax colimits and free fibrations in ∞-categories. *Doc. Math.*, 22:1225–1266, 2017. (Cited on page 2.)

[Gle58] Andrew M. Gleason. Projective topological spaces. *Illinois J. Math.*, 2:482–489, 1958. (Cited on page 2.)

[GLP26] David Gepner, Sil Linskens, and Luca Pol. Global 2-rings and genuine refinements, 2026. arXiv:2407.05124. (Cited on pages 4, 5, and 6.)

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. (Cited on pages 2 and 8.)

[Har25] Oscar Bendix Harr. Compact sheaves on a locally compact space. *Proc. Amer. Math. Soc.*, 153(1):55–68, 2025. (Cited on page 3.)

[HHL+25] Peter J. Haine, Tim Holzschuh, Marcin Lara, Catrin Mair, Louis Martini, and Sebastian Wolf with an appendix by Bogdan Zavyalov. The condensed homotopy type of a scheme, 2025. arXiv:2510.07443. (Cited on pages 3 and 13.)

[HHLN23] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. Lax monoidal adjunctions, two-variable fibrations and the calculus of mates. *Proc. Lond. Math. Soc. (3)*, 127(4):889–957, 2023. (Cited on page 6.)

[Hil24] Kaif Hilman. Parametrised presentability over orbital categories. *Appl. Categ. Structures*, 32(3):Paper No. 15, 53, 2024. (Cited on page 11.)

[HSSS21] Marc Hoyois, Pavel Safronov, Sarah Scherotzke, and Nicolò Sibilla. The categorified Grothendieck-Riemann-Roch theorem. *Compos. Math.*, 157(1):154–214, 2021. (Cited on pages 2, 3, and 8.)

[LNP25] Sil Linskens, Denis Nardin, and Luca Pol. Global homotopy theory via partially lax limits. *Geometry and Topology*, 29(3):1345–1440, May 2025. (Cited on pages 5, 9, 11, and 12.)

[Lur18] Jacob Lurie. Spectral Algebraic Geometry. *Available from the author's webpage*, 2018. Online manuscript. (Cited on pages 1, 2, 4, 7, and 11.)

[Mai26] Catrin Mair. On Galois categories and condensed contractible schemes, 2026. arXiv:2605.10358. (Cited on pages 3 and 13.)

[Nik16] Thomas Nikolaus. Stable ∞-operads and the Multiplicative Yoneda Lemma, 2016. arXiv:1608.02901. (Cited on page 9.)

[Ram26] Maxime Ramzi. Locally rigid ∞-categories, 2026. arXiv:2410.21524. (Cited on pages 2 and 8.)

[RZ26] Nima Rasekh and Qi Zhu. Fractured Structures in Condensed Mathematics, 2026. arXiv:2603.09618. (Cited on pages 3, 4, and 12.)
[Sch26a] Peter Scholze. Lectures on analytic geometry, 2026. arXiv:2605.03655. (Cited on pages 3 and 4.)
[Sch26b] Peter Scholze. Lectures on condensed mathematics, 2026. arXiv:2605.03658. (Cited on page 3.)
[vDdB26] Remy van Dobben de Bruyn. A condensed proof of the pro-étale and étale exodromy theorems, 2026. arXiv:2605.22499. (Cited on pages 3, 4, and 13.)
[Wol22] Sebastian Wolf. The pro-étale topos as a category of pyknotic presheaves. *Doc. Math.*, 27:2067–2106, 2022. (Cited on page 3.)

F. Neugebauer, Mathematisches Institut, University of Bonn, Germany
E-mail address: fabio@math.uni-bonn.de

Q. Zhu, Max Planck Institute for Mathematics, Bonn, Germany
E-mail address: qzhu@mpim-bonn.mpg.de