Formal Group Laws and Stable Homotopy Theory over Profinite Abelian Groups

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Abstract. We generalize the theory of complex oriented cohomology theories to the setting of A-equivariant homotopy theory, for a compactly metrizable abelian group A. A complex orientation will allow for the computation of the (co)-homology of A-equivariant Grassmannians. The A-equivariant complex bordism spectrum carries the universal A-equivariant orientation.

Our study of complex oriented A-spectra will lead naturally to a definition of A-equivariant formal group law. We identify the homotopy groups of the A-equivariant complex bordism spectrum with the A-equivariant Lazard ring.

We factor HKR's chromatic character map via geometric fixed points through the $\mathbb{Z}_p^n$-equivariant Borel Lubin-Tate theory.

Contents

1. Introduction

1.1. **The Classical Story.** To every complex oriented cohomology theory one can assign a formal group law. This connection between stable homotopy theory and formal group laws is extremely firm: While the complex bordism spectrum **MU** is the initial example of a complex oriented spectrum, it is a theorem of Quillen that the formal group law associated to **MU** is the initial example of a formal group law. The theory of complex orientations and formal group laws has deep applications in stable homotopy theory such as the classification of thick subcategories of finite spectra by Devinatz, Hopkins and Smith [DHS88].

1.2. **Motivation.** In this thesis, we generalize the above classical story to a broader setup. We allow the appearing spaces, as well as their cohomology theories, to carry the action of a compactly metrizable abelian group A. Spaces with an A-action assemble into the ∞-category of genuine A-spaces $\mathcal{S}_A$. Cohomology theories for A-spaces form the ∞-category of genuine A-spectra. In this thesis we concern ourselves with the following natural questions:

- For which A-equivariant cohomology theories can we compute the cohomology of the classifying space of A-equivariant complex vector bundles?
- Can we find universal properties of the A-equivariant complex bordism theory and its associated genuine A-spectrum $\mathbf{MU}_A$?
- Can the homotopy groups of the A-equivariant complex bordism spectrum $\mathbf{MU}_A$ be described analogously to their classical counterparts?

At first glance, the generality permitted by an arbitrary compactly metrizable abelian group A may appear excessive—particularly given that the questions posed above have already been answered affirmatively in the case where A is a compact abelian Lie group; see [CGK02] and [Hau22]. However, this flexibility is in fact well motivated. The HKR character map, originally introduced in [HKR00], is a celebrated and well studied construction in chromatic homotopy theory, see [Rez06], [Sta13b], [BS17], and [Lur19].

We show that the HKR character map arises as the comparison map between Borel Lubin-Tate theory and its $\mathbb{Z}_p^n$-equivariant geometric fixed points. In this sense, the HKR character map foreshadows a well-behaved $\mathbb{Z}_p^n$-equivariant homotopy theory. Here, $\mathbb{Z}_p^n$ denotes the n-fold product of the topological group of p-adic integers $\mathbb{Z}_p$.

On the one hand, our interpretation of the HKR character map can shed light on its good functoriality properties. Indeed, we prove that the HKR character map is the effect of a symmetric monoidal left adjoint functor on mapping spectra. On the other hand, $\mathbb{Z}_p^n$-equivariant Borel Lubin-Tate theory is equivariantly complex orientable. Hence, it might be fruitful to study the HKR character map from the perspective of $\mathbb{Z}_p^n$-equivariant formal group laws.

1.3. **The Setup.** A substantial portion of the thesis is devoted to recalling and developing equivariant homotopy theory for general compact metrizable groups A, without assuming A to be abelian. To address the motivating questions posed above, we later specialize to the case where A is abelian.

1.3.1. *The Symmetry Group.* A topological group A is compactly metrizable if and only if A is isomorphic to an inverse limit

$$\varprojlim (A_0 \twoheadleftarrow A_1 \twoheadleftarrow A_2 \twoheadleftarrow A_3 \twoheadleftarrow A_4 \twoheadleftarrow \cdots)$$

with each A_n a compact Lie group. Examples of compactly metrizable groups are first-countable, a.k.a. light, profinite groups, such as the p-adic integers or the Morava stabilizer group. Other examples of compactly metrizable groups include solenoids, i.e. countable inverse limits of tori, or countable products of compact Lie groups. Let us fix such a compactly metrizable group A for the rest of this introduction.

1.3.2. *Genuine A-Equivariant Spaces and Spectra.* Fausk [Fau08] defined the ∞-category of A-spaces $\mathcal{S}_A$ via a model structure on the category of spaces with A-action. A generalized version of Elmendorf's theorem states that the fixed point functors associated to certain subgroups of A exhibit the ∞-category of A-spaces as category of presheaves on an orbit category. Using the formula for A as an inverse limit $A \cong \varprojlim_n A_n$ we obtain a formula for A-spaces $\mathcal{S}_A \simeq \varprojlim_n \mathcal{S}_{A_n}$ as an inverse limit in Cat_∞.

In [Fau08], the author constructed the ∞-category of A-spectra Sp_A by passing to the underlying ∞-category of a suitable model category of orthogonal A-spectra. We show that the fixed point functors induce an equivalence $\mathrm{Sp}_A \simeq \varprojlim_n \mathrm{Sp}_{A_n}$ in Cat_∞. In the case that A is profinite, this result is due to [BBB24]. Generalizing a result of [GM23], we prove that the symmetric monoidal ∞-category of A-spectra enjoys the following universal property: The suspension functor $\Sigma^\infty : \mathcal{S}_{A,*} \to \mathrm{Sp}_A$ is the initial morphism in $\mathrm{CAlg}(\mathrm{Pr}^L)$ out of pointed A-spaces sending all representations spheres to tensor-invertible objects.

For a closed subgroup $B \leq A$, there are two notions of fixed point functors. The *categorical fixed point functor* $(-)^B \colon \mathrm{Sp}_A \to \mathrm{Sp}$ is corepresented by the orbit spectrum $\Sigma^\infty_+ A/B$. On the other hand, the *geometric fixed point functor* $\Phi^B_A \colon \mathrm{Sp}_A \to \mathrm{Sp}$ satisfies a compatibility relation with suspension spectra:

$$\Phi^B_A \circ \Sigma^\infty \simeq \Sigma^\infty \circ (-)^B.$$

The associated *B-fixed homotopy groups*

$$\pi^B_* := \pi_* \circ (-)^B \colon \mathrm{Sp}_A \to \mathrm{Ab}^{\mathrm{gr}}$$

form a jointly conservative family of functors to graded abelian groups, as B ranges over those subgroups of A, which are "well-behaved" in the sense of Definition 2.3.1.

1.3.3. *Connection to Global Spectra.* We construct a symmetric monoidal left adjoint functor

$$\mathrm{Res}_A \colon \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_A$$

from Schwede's category of global spectra to the category of genuine A-spectra, extending Schwede's construction to compactly metrizable groups. For a global spectrum $\mathcal{X}$, the global structure induces comparison maps

$$\pi^{A_n}_* \left(\mathrm{Res}_{A_n}(\mathcal{X})\right) \to \pi^{A_{n+1}}_* \left(\mathrm{Res}_{A_{n+1}}(\mathcal{X})\right)$$

along the inverse system of group homomorphisms $A_{n+1} \to A_n$. We construct an isomorphism

$$\pi^A_* \left(\mathrm{Res}_A(\mathcal{X})\right) \cong \mathrm{colim}_n\, \pi^{A_n}_* \left(\mathcal{X}_{A_n}\right), \tag{1}$$

naturally in $\mathcal{X} \in \mathrm{Sp}_{\mathrm{gl}}$. In addition, we enhance the point-set Thom space construction to a symmetric monoidal left adjoint Thom spectrum functor $\mathrm{Th}_A : \mathcal{S}_{/\,\mathbf{BOP}_A} \to \mathrm{Sp}_A$. The construction comes with equivalences

$$\mathrm{Th}_A \circ \mathrm{Infl}^A_{A_n} \simeq \mathrm{Infl}^A_{A_n} \circ \mathrm{Th}_{A_n} \qquad \text{and} \qquad \mathrm{Th}_A \circ \mathrm{Res}_A \simeq \mathrm{Res}_A \circ \mathrm{Th}_{\mathrm{gl}},$$

where $\mathrm{Th}_{\mathrm{gl}}$ is Schwede's global symmetric monoidal Thom spectrum functor.

1.4. ***A*-Equivariant Chromatic Homotopy Theory.** When A is a compact abelian Lie group, Cole, Kriz, and Greenlees [CGK00] introduced the notion of a *complex oriented A-spectrum*, along with the associated *A-equivariant formal group law*. These definitions extend verbatim to the broader class of abelian groups considered in Setup 1.3.

In this generalized setup, we extend the computations from [CGK02] to describe the A-equivariant (co)homology rings

$$E^*_A(\mathrm{B}_A(U(d))) \qquad \text{and} \qquad E^A_*(\mathbf{MU}_A),$$

where $\mathrm{B}_A(U(d))$ denotes the classifying space for A-equivariant d-dimensional complex vector bundles, and $\mathbf{MU}_A$ is the A-equivariant complex bordism spectrum. These formulas hold for any complex oriented A-spectrum E.

We then prove that the A-equivariant complex bordism spectrum $\mathbf{MU}_A$ is the initial example of a complex oriented commutative homotopy ring A-spectrum.

Beyond this, our main results are the following: Assume an abelian topological group A is isomorphic to an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit of compact Lie groups A_n. Then, we show

Theorem (Equivariant Lazard's Theorem)**.** *The universal A-equivariant formal group law exists and is defined over the colimit of equivariant Lazard rings*

$$L_A := \mathrm{colim}_n\, L_{A_n}.$$

Theorem (Equivariant Quillen's Theorem)**.** *The graded ring homomorphism*

$$L_A \xrightarrow{\cong} \pi_*^A\, \mathbf{MU}_A$$

classifying the A-equivariant formal group law associated to the universal complex orientation of $\mathbf{MU}_A$ is an isomorphism.

The equivariant Quillen's theorem is known for compact abelian Lie groups due to [Hau22]. The global structure of the complex bordism spectrum and our formula (1) allows us to pass to limits with respect to the symmetry group:

$$L_A \simeq \mathrm{colim}_n\, L_{A_n} \simeq \mathrm{colim}_n\, \pi_*^{A_n}(\mathbf{MU}_{A_n}) \simeq \pi_*^A(\mathbf{MU}_A).$$

1.5. **Factorization of the HKR-Character Map.** Let E be a p-local height n Lubin-Tate spectrum. The HKR character map is a tool to compute the rationalized E-cohomology ring of the classifying space $\mathrm{B}G$, whenever G is a finite group. If X is a finite G-space, then the HKR character map

$$E^*(X_{hG}) \xrightarrow{\mathrm{HKR}} L^*(E) \otimes_{E^*} E^*(\mathrm{Fix}_{\mathbb{Z}_p^n}(X, G))^G \tag{2}$$

becomes an isomorphism after base change to a certain E^*-algebra $L^*(E)$. In [HKR00], the G-space $\mathrm{Fix}_{\mathbb{Z}_p^n}(X, G)$ is defined via an explicit formula, while the $\mathbb{Q}$-algebra $L^*(E)$ is shown to have a certain universal property as an E^*-algebra. In the case that X is a point, the G-space $\mathrm{Fix}_{\mathbb{Z}_p^n}(X, G)$ identifies with the set of group homomorphisms $\hom_{\mathrm{Grp}}(\mathbb{Z}_p^n, G)$ equipped with the conjugation G-action and we obtain an algebraic formula for $L^*(E) \otimes_{E^*} E^*(\mathrm{B}G)$.

The relation to $\mathbb{Z}_p^n$-equivariant homotopy theory comes from the following two observations:

(1) Consider the $\mathbb{Z}_p^n$-equivariant Borel theory $E^{b\mathbb{Z}_p^n}$. The homotopy ring $\mathbb{Z}_p^n$-spectrum $E^{b\mathbb{Z}_p^n}$ inherits a complex orientation from the Lubin-Tate spectrum E. The general formula for geometric fixed points of complex oriented spectra, see Corollary 3.5.9, provides a preferred graded ring isomorphism

$$\pi_{-*}\left(\Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})\right) \to L^*(E).$$

(2) For a finite group G, the G-restriction functor $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$ from global spaces admits a left adjoint $(-)_{/\!\!/G} : \mathcal{S}_G \to \mathcal{S}_{\mathrm{gl}}$. The global orbit functor $(-)_{/\!\!/G}$ lifts the homotopy orbit functor, in the sense that for a G-space X we have an equivalence between its homotopy orbits X_{hG} and the underlying space $\mathrm{Res}_e(X_{/\!\!/G})$. There is a way to "interchange" the process of taking (global/homotopy) G-orbits and $\mathbb{Z}_p^n$-fixed points: We construct an explicit equivalence

$$I : \mathrm{Fix}_{\mathbb{Z}_p^n}(G, X)_{hG} \xrightarrow{\simeq} (\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!\!/G}))^{\mathbb{Z}_p^n}$$

from the homotopy orbits of the G-space $\mathrm{Fix}_{\mathbb{Z}_p^n}(G,X)$ to the $\mathbb{Z}_p^n$-fixed points of the $\mathbb{Z}_p^n$-space $\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!/G})$.

In Theorem 4.0.1, we prove that the HKR-character map factors as follows

$$\begin{array}{ccccc}
E(X_{hG}) & & \xrightarrow{\quad\mathrm{HKR}\quad} & & L^*(E)\otimes_{E^*} E(\mathrm{Fix}_{\mathbb{Z}_p^n}(X,G))^G \\
\downarrow{\cong} & & & & \downarrow{\cong} \\
E^b(X_{/\!/G}) & \xrightarrow{\mathrm{Res}_{\mathbb{Z}_p^n}} & E^{b\mathbb{Z}_p^n}(\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!/G})) & \xrightarrow{\Phi^{\mathbb{Z}_p^n}} & \Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})(\mathrm{Fix}_{\mathbb{Z}_p^n}(X,G)_{hG})
\end{array}$$

From left to right, the nodes of the diagram denote unreduced cohomology with coefficients in the following cohomology theories (in the diagram we abbreviated E^* with E for all cohomology theories): the Lubin-Tate theory spectrum $E\in\mathrm{Sp}$, the associated global Borel theory $E^b\in\mathrm{Sp}_{\mathrm{gl}}$, the $\mathbb{Z}_p^n$-equivariant Borel theory $E^{b\mathbb{Z}_p^n}:=\mathrm{Res}_{\mathbb{Z}_p^n}(E^b)\in\mathrm{Sp}_{\mathbb{Z}_p^n}$ and the geometric fixed point spectrum $\Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})\in\mathrm{Sp}$.

Let us explain the maps in the diagram in detail:

- The left vertical isomorphism in the diagram comes from the adjunction
$$\mathrm{Res}_e\colon \mathrm{Sp}_{\mathrm{gl}}\rightleftarrows\mathrm{Sp}:(-)^b \qquad \text{and the equivalence} \qquad X_{hG}\simeq\mathrm{Res}_e\left(X_{/\!/G}\right)$$
of spaces. Here $(-)_{/\!/G}:\mathrm{Sp}_G\to\mathrm{Sp}_{\mathrm{gl}}$ is the left adjoint of G-restriction.
- The right hand vertical isomorphism is induced by the ring map $E\to\Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})$.
- The left lower horizontal arrow is the effect of the restriction functor $\mathrm{Res}_{\mathbb{Z}_p^n}:\mathrm{Sp}_{\mathrm{gl}}\to\mathrm{Sp}_{\mathbb{Z}_p^n}$ on mapping spectra
$$\mathrm{Res}_{\mathbb{Z}_p^n}:\ \mathrm{map}(\Sigma^\infty_+X_{/\!/G},E^b)\to\mathrm{map}(\Sigma^\infty_+\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!/G}),E^{b\mathbb{Z}_p^n}).$$
- The right lower horizontal arrow is the effect of the geometric fixed point functor $\Phi^{\mathbb{Z}_p^n}:\mathrm{Sp}_{\mathbb{Z}_p^n}\to\mathrm{Sp}$ on mapping spectra
$$\Phi^{\mathbb{Z}_p^n}:\mathrm{map}(\Sigma^\infty_+\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!/G}),E^{b\mathbb{Z}_p^n})\to\mathrm{map}(\Sigma^\infty_+\mathrm{Fix}_{\mathbb{Z}_p^n}(X,G),\Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})).$$

1.6. **Organization of the Thesis.** In Section 2, we dive into genuine equivariant homotopy theory with a compactly metrizable symmetry group. In Section 3, we study complex oriented A-spectra and A-equivariant formal group laws, for compactly metrizable abelian groups. In Section 4, we factor the HKR character map applied to a G-space X through Borel equivariant cohomology of the genuine $\mathbb{Z}_p^n$-space $\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!/G})$.

1.6.1. *Summary of Section 2: Pro Compact Lie Equivariant Homotopy Theory.* We establish a continuous version for the category of G-spaces and G-spectra in the sense of [BBB24], for any compactly metrizable group G. In Proposition 2.6.10, we generalize the universal property of the category of genuine G-spectra with respect to inverting representation spheres from compact Lie groups to compactly metrizable groups.

In Section 2.5.1, we study classifying spaces of G-equivariant L-principal bundles $\mathrm{B}_G(L)$. As structure group we allow for any compact Lie group L. We may choose an isomorphism $G\to\varprojlim G_n$ to an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms $G_{n-1}\twoheadleftarrow G_n$ of compact Lie groups G_n. In that situation, we can derive a colimit formula

$$\operatorname*{colim}_{n\in\mathbb{N}}\mathrm{Infl}_{G_n}^G(\mathrm{B}_{G_n}(L))\xrightarrow{\simeq}\mathrm{B}_G(L)$$

for the G-equivariant classifying space.

For any compactly metrizable group G, we construct and establish properties of the restriction functor $\mathrm{Res}_G:\mathrm{Sp}_{\mathrm{gl}}\to\mathrm{Sp}_G$ from global spectra, as well as of the G-equivariant Thom spectrum functor $\mathrm{Th}_G : \mathcal{S}_{/\mathbf{BOP}_G} \to \mathrm{Sp}_G$. The unstable version $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$ will send a global classifying space $\mathrm{B}_{\mathrm{gl}}L$ of a compact Lie group L to the classifying space $\mathrm{B}_G(L)$ of G-equivariant L-principle bundles, see Construction 2.7.1. The G-equivariant Thom spectrum functor will be symmetric monoidal and natural in the compactly metrizable group G, see Theorem 2.8.11.

Finally, in Section 2.9, we discuss examples of G-equivariant (ring)-spectra, such as Borel spectra, the G-equivariant complex K-theory spectrum and the G-equivariant complex bordism spectrum. For any compact metrizable group G, we discuss both telescoping formulas and Thom classes for G-equivariant complex bordism spectra.

1.6.2. *Summary of Section 3: Universality of Equivariant Complex Bordism.* We fix an abelian topological group A with discrete and countable Pontryagin dual A^*, e.g. the p-adic integers $A = \mathbb{Z}_p$ or a solenoid $A = \mathbb{T}_p$. Equivalently, A is an abelian compactly metrizable group. We choose morphisms $\varphi_n : A \to A_n$ exhibiting the topological group A as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms $A_{n-1} \twoheadleftarrow A_n$ of compact abelian Lie groups A_n.

Our Definition 3.1.4 of a complex oriented A-spectrum E is essentially the definition from [Col96], adapted to our setup. A complex orientation

$$x(\epsilon) \in E_A^*(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$$

enables an explicit computation of the E-cohomology of A-equivariant complex projective spaces $\mathbb{CP}(V)$, such as the classifying space $\mathbb{CP}_A^\infty := \mathrm{B}_A(U(1))$ of A-equivariant complex line bundles, see Proposition 3.1.17. The structure and properties of the unreduced cohomology $E_A^*(\mathbb{CP}_A^\infty)$ of equivariant projective space motivates the definition of an A-equivariant formal group law over $\pi_*^A(E)$. As discussed in Section 3.2, the notion of an A-equivariant formal group law is algebraic and defines an invariant of complex oriented A-spectra.

In Section 3.3, we discuss co-restriction of equivariant formal group laws along continuous group homomorphisms $\varphi : A \to B$ of compactly metrizable abelian groups. If an A-equivariant formal group law F was associated to a complex oriented A-spectrum E, then the co-restriction $\varphi_* F$ is associated to the coinduced/fixed complex oriented B-spectrum $\varphi_* B$, which carries the pushforward complex orientation $\varphi_*(x(\epsilon))$, see Section 3.3.1. In Proposition 3.4.3, we prove that the co-restrictions exhibit the category of A-equivariant formal group laws as an inverse limit of the categories of A_n-equivariant formal group laws. This allows us to deduce the A-equivariant Lazard theorem 3.4.5 from the abelian compact Lie group case [CGK00].

In Section 3.5, we compute the E-(co)homology of the classifying spaces $\mathrm{B}_A(U(d))$ of A-equivariant complex vector bundles. In addition, we establish the Thom isomorphism in E-(co)homology for A-equivariant complex vector bundles. Here, E is a complex oriented A-spectrum. For $\varphi_n : A \to A_n$ the projections of the inverse limit, the A_n-spectrum $(\varphi_n)_* E$ carries the pushforward complex orientation. The abelian compact Lie group theory [CGK02] of complex oriented A_n-spectra allows for an explicit computation of the $(\varphi_n)_* E$-homology groups of $\mathrm{B}_{A_n}(U(n))$. Taking the colimit along the inflations $\mathrm{Infl}_{A_n}^{A_{n+1}}(-)$ allows us to deduce the E-homology of $\mathrm{B}_A(U(d))$ from that computation, see Proposition 3.5.2.

In Section 3.6, we show that the complex bordism spectrum $\mathbf{MU}_A$ is the initial example of a complex oriented A-spectrum. Moreover, we compute the $\pi_*^A(E)$-algebra $E_*^A(\mathbf{MU}_A)$, i.e. the E-homology of the complex bordism spectrum, whenever E is a complex oriented A-spectrum. Lastly, having established topological universality of the complex bordism spectrum, we are in a position to prove the A-equivariant version of Quillen's theorem, see Theorem 3.7.1.

1.6.3. *Summary of Section 4: Factorization of the HKR-Character Map.* The overall goal of Section 4 is proving the factorization of the HKR character map as in Theorem 4.0.1. In the subsection on Formal Loop Spaces, we recall the construction of the G-space $\mathrm{Fix}_A(G,X)$, associated to an abelian complex metrizable group A and a G-space X. We then construct equivalences

$$\mathrm{Fix}_A(G,X)_{hG} \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!/G}) \xrightarrow{\simeq} \big(\mathrm{Res}_A(X_{/\!/G})\big)^A$$

naturally in the G-space X, interchanging A-fixed points with G-orbits.

In Section 4.2, we check that a complex orientation of a homotopy ring spectrum E induces an A-equivariant complex orientation of the associated Borel A-spectrum E^{bA}. Moreover, we relate the Euler class associated to a character $\alpha : A \to U(1)$ to the Chern class associated to the line bundle $\mathrm{B}\alpha : \mathrm{B}A \to \mathrm{B}U(1) = \mathbb{CP}^\infty$. The geometric fixed points of a complex oriented spectrum are obtained from the categorical fixed points by inverting Euler classes, see Corollary 3.5.9. In conclusion, we obtain a formula for the geometric fixed point ring spectrum $\Phi^A(E^{bA})$, entirely in terms of the complex oriented spectrum E.

In particular, applying this formula to a height n Lubin-Tate theory spectrum E and an n-fold product of the p-adic integers $A = \mathbb{Z}_p^n$, we identify the homotopy ring of the geometric fixed point spectrum $\Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})$ with the E^*-algebra $L^*(E)$ from the HKR character map.

Finally, we conclude Section 4 by giving a proof of the factorization of the HKR character map stated in Theorem 4.0.1.

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2. Pro Compact Lie Equivariant Homotopy Theory

2.1. Compactly Metrizable Groups as Pro Compact Lie Groups.

Definition 2.1.1. For a topological group G, we write $G \in \mathrm{Grp}(\mathrm{CptMet})$ if G is isomorphic to some $\mathbb{N}^{\mathrm{op}}$ indexed inverse limit

$$\varprojlim \Big(G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_3} G_3 \xleftarrow{\phi_4} \cdots \Big) \tag{3}$$

in the category of topological groups, with each $G_n \in \mathrm{CptLie}$ a compact Lie group and $\phi_n : G_n \to G_{n-1}$ a surjective, continuous group homomorphism. We view $\mathrm{Grp}(\mathrm{CptMet})$ as a topological category with morphism spaces consisting of continuous group homomorphisms

$$\underline{\mathrm{Hom}}_{\mathrm{Grp}}(G,L) \subseteq \underline{\mathrm{Hom}}_{\mathrm{Top}}(G,L)$$

equipped with the compact-open topology. $\mathrm{CptLie} \subseteq \mathrm{Grp}(\mathrm{CptMet})$ denotes the full topological subcategory spanned by compact Lie groups.

Remark 2.1.2. The Peter-Weyl theorem for compact Hausdorff groups implies that a topological group G is in $\mathrm{Grp}(\mathrm{CptMet})$ if and only if G is compact and metrizable, see [BTD85, Theorem III.3.1.]. A compact Hausdorff group G is metrizable if and only if it is first-countable if and only if it is second-countable, see [BK96, page 14].

Lemma 2.1.3. Let G be an inverse limit of compact Hausdorff topological groups G_n along surjective homomorphisms

$$G := \varprojlim_n \left(G_0 \twoheadleftarrow G_1 \twoheadleftarrow G_2 \twoheadleftarrow G_3 \twoheadleftarrow \cdots \right).$$

If the topological group G admits the structure of a Lie group, then the inverse limit stabilizes, i.e. there exists $m \geq 0$, such that $G_{n+1} \to G_n$ is a homeomorphism for all $n \geq m$.

Proof. By [RZ00, Lemma 1.1.5.], the continuous homomorphism $G \to G_n$ is surjective. We conclude that each G_n is a Lie group. Without loss of generality, G_0 already has the maximal dimension among all the G_n. When we write $N_n := \ker(G \to G_n)$, then $N_0/N_n = \ker(G_n \to G_0)$ is a finite, discrete group. Moreover, by [RZ00, Corollary 1.1.18.], the canonical map $N_0 \to \varprojlim N_0/N_n$ is a homeomorphism. Thus, $N_0 \leq G$ is both a profinite group, as well as a compact Lie group. A totally disconnected, and locally connected space is discrete. By compactness, N_0 is finite. Thus, we may choose $m \geq 0$, so that $N_0 = N_n$ for all $n \geq m$. □

Theorem 2.1.4 (Cartan's Theorem)**.** *Suppose morphisms $\varphi_n : G \to G_n$ exhibit a topological group G as an inverse limit $G \cong \varprojlim G_n$ of compact Lie groups G_n along surjections $G_{n-1} \twoheadleftarrow G_n$. Let L be a Lie group. Precomposition by the projections $\varphi_n : G \to G_n$ induces a homeomorphism*

$$\mathrm{colim}_n\, \underline{\mathrm{Hom}}_{\mathrm{Grp}}(G_n, L) \to \underline{\mathrm{Hom}}_{\mathrm{Grp}}(G, L)$$

for the compact-open topology on the set of continuous group homomorphisms.

Proof. The colimit is taken along open embeddings of unions of connected components, see [Sch18, Proposition A.25.]. To see that the map is surjective, we need to see that any continuous homomorphism $\alpha : G \to L$ factors through some G_n. By Cartan's theorem, the closed subgroup $G/\ker(\alpha) \leq L$ of the Lie group L is a Lie group itself. By [RZ00, Lem. 1.1.5.], the map $G/\ker(\alpha) \to \varprojlim G/(\ker(\varphi_n)\ker(\alpha))$ is surjective. Thus, the the latter inverse limit must produce a compact Lie group and, therefore, stabilizes by Lemma 2.1.3. We find $n \geq 0$ with $\ker(\varphi_n)\ker(\alpha) \leq \ker(\alpha)$, so that α factors through G_n. □

Example 2.1.5. If we write $\mathbb{T}$ for the 1-dimensional unitary group and $(-)^p : \mathbb{T} \to \mathbb{T}$ for the p-th power homomorphism, then the *solenoid*

$$\mathbb{T}_p := \varprojlim \left(\mathbb{T} \xleftarrow{(-)^p} \mathbb{T} \xleftarrow{(-)^p} \mathbb{T} \xleftarrow{(-)^p} \mathbb{T} \leftarrow \cdots \right)$$

is an inverse limit along surjective homomorphisms of compact Lie groups. From Cartan's Theorem 2.1.4, we conclude that the Pontryagin dual $\mathbb{T}_p^* := \underline{\mathrm{Hom}}_{\mathrm{Grp}}(\mathbb{T}_p, \mathbb{T})$ is isomorphic to the discrete topological group

$$\mathbb{Z}[p^{-1}] := \mathrm{colim}(\mathbb{Z} \xrightarrow{p\cdot(-)} \mathbb{Z} \xrightarrow{p\cdot(-)} \mathbb{Z} \xrightarrow{p\cdot(-)} \mathbb{Z} \to \cdots).$$

Example 2.1.6. Pontryagin duality [Pon34]

$$G \mapsto G^* := \underline{\mathrm{Hom}}_{\mathrm{Grp}}(G, \mathbb{T})$$

defines an equivalence from the category of compact abelian Lie groups to the opposite of the category of finitely generated abelian groups.

It follows from Theorem 2.1.4 that the Pontryagin dual $G^* := \underline{\mathrm{Hom}}_{\mathrm{Grp}}(G, \mathbb{T})$ of any compactly metrizable group G is isomorphic to an $\mathbb{N}$-indexed colimit of finitely generated abelian groups, equipped with the discrete topology. As any countable abelian group is isomorphic to an $\mathbb{N}$-indexed colimit of finitely generated abelian groups, Pontryagin duality defines an equivalence from the category of compactly metrizable abelian groups to the opposite of the category of countable discrete abelian groups, see [Mor77].

2.2. Functoriality of G-spaces (G-spectra) in the Compactly Metrizable Group G.

Construction 2.2.1. By Theorem 2.1.4, the topologically-enriched restricted Yoneda embedding

$$\mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CptLie}, \mathrm{Top}), \qquad G \mapsto \underline{\mathrm{Hom}}_{\mathrm{Grp}}(G, -)$$

factors through bi-fibrant objects in the projective Quillen model structure. By Theorem 2.1.4, the induced[1] functor on underlying ∞-categories, factors through the ∞-categorial Ind-category

$$\mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \to \mathrm{Ind}(\mathrm{CptLie}^{\mathrm{op}}) \subseteq \mathrm{Fun}(\mathrm{CptLie}, \mathcal{S}). \tag{4}$$

Here we use the same notation for a topological category and its homotopy-coherent nerve.

Definition 2.2.2. Passing to opposite categories in Equation (4) yields the *pro-analogue functor* of ∞-categories

$$\nu : \mathrm{Grp}(\mathrm{CptMet}) \to \mathrm{Pro}(\mathrm{CptLie}).$$

Unraveling the construction, if a topological group G is an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective continuous homomorphisms of compact Lie groups $G_{n-1} \twoheadleftarrow G_n$, then

$$\nu(G) \to \varprojlim \nu(G_n) = \varprojlim G_n \tag{5}$$

is an equivalence of pro-objects.

Construction 2.2.3. We construct the genuine G-spaces (G-spectra) functors

$$\mathcal{S}_\bullet : \quad \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \to \mathrm{Pr}^L, \qquad G \mapsto \mathcal{S}_G \tag{6}$$

$$\mathrm{Sp}_\bullet : \quad \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \to \mathrm{Pr}^L, \qquad G \mapsto \mathrm{Sp}_G \tag{7}$$

by uniquely extending the functors

$$\mathrm{CptLie}^{\mathrm{op}} \to \mathrm{Pr}^L, \qquad G \mapsto \mathcal{S}_G \tag{8}$$

$$\mathrm{CptLie}^{\mathrm{op}} \to \mathrm{Pr}^L, \qquad G \mapsto \mathrm{Sp}_G \tag{9}$$

from [LNP25] in a filtered colimit preserving way to the Ind-category

$$\mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} := \mathrm{Ind}(\mathrm{CptLie}^{\mathrm{op}}).$$

In the compact Lie group case, the above functoriality of G-spaces is induced by restriction/inflation along continuous homomorphisms of compact Lie groups. We will see in Section 2.4.2 that the same is true for morphisms of compactly metrizable groups.

Definition 2.2.4. For $G \in \mathrm{Grp}(\mathrm{CptMet})$, we call $\mathcal{S}_G := \mathcal{S}_{\nu(G)} \in \mathrm{Pr}^L$ the category of *genuine G-spaces* and $\mathrm{Sp}_G := \mathrm{Sp}_{\nu(G)} \in \mathrm{Pr}^L$ the category of *genuine G-spectra*.

In Section 2.4.2 and Corollary 2.6.15, we construct preferred equivalences from the ∞-category of genuine G-spaces, respectively, genuine G-spectra to the underlying ∞-category of Fausk's model categories from [Fau08].

Remark 2.2.5. Fausk's constructions [Fau08] apply to general compact Hausdorff groups. By the Peter–Weyl theorem [BTD85, III.3.1.], a topological group G is compact Hausdorff if and only if it can be expressed as an I^{op}-indexed limit of compact Lie groups for some filtered poset I. The diligent reader will have little difficulty extending most of the results in Section 2 to this more general setting. However, later in our discussion of equivariant formal group laws, the choice of a complete flag is crucial for inductive arguments. Choosing such a flag for G requires that the poset I be countable, or equivalently, that the topological group G be metrizable. Therefore, we continue to assume that all compact Hausdorff groups under consideration are metrizable.

Convention 2.2.6. For a continuous group homomorphism $\varphi : K \to L$ of compactly metrizable groups $K, L \in \mathrm{Grp}(\mathrm{CptMet})$, we have an induced functor $\varphi^* : \mathcal{S}_L \to \mathcal{S}_K$.

[1] The canonical morphism $N^{hc}\left(\mathrm{Fun}_{\mathrm{Top}}(\mathrm{CptLie}, \mathrm{Top})^{\circ}_{\mathrm{proj}}\right) \to \mathrm{Fun}(N^{hc}(\mathrm{CptLie}), \mathcal{S})$ is an equivalence of ∞-categories by [Lur09, Prop. 4.2.4.4.].

(1) If φ is surjective, we call the functor φ^* *inflation* and write $\mathrm{Infl}_L^K := \varphi^*$. The right adjoint of inflation is called $\ker(\phi)$-*fixed points* and denoted $\varphi_* : \mathcal{S}_K \to \mathcal{S}_L$.
(2) If φ is injective, we call the functor φ^* *restriction* and write $\mathrm{Res}_K^L := \varphi^*$. The right adjoint of restriction is called *co-induction* and denoted $\varphi_* : \mathcal{S}_K \to \mathcal{S}_L$.

We introduce the same notation on the level of pointed objects $\mathcal{S}_{K,*}$ and spectra Sp_K.

2.3. **Genuine G-Spaces.** Unraveling the construction, if a topological group G is an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms compact Lie groups $G_{n-1} \twoheadleftarrow G_n$, then the inflations induce an equivalence

$$\operatorname{colim}^L \left(\mathcal{S}_{G_0} \xhookrightarrow{\mathrm{Infl}_{G_0}^{G_1}} \mathcal{S}_{G_1} \xhookrightarrow{\mathrm{Infl}_{G_1}^{G_2}} \mathcal{S}_{G_2} \xhookrightarrow{\mathrm{Infl}_{G_2}^{G_3}} \mathcal{S}_{G_3} \xhookrightarrow{\mathrm{Infl}_{G_3}^{G_4}} \mathcal{S}_{G_4} \hookrightarrow \cdots \right) \to \mathcal{S}_G \tag{10}$$

of ∞-categories, where the colimit is taken in Pr^L. Thus, Yoneda extending the composite

$$\mathrm{Orb}_{G_n} \to \mathcal{S}_{G_n} \xrightarrow{\mathrm{Infl}_{G_n}^G} \mathcal{S}_G$$

exhibits $\mathcal{S}_G$ as presheaves on $\operatorname{colim}_n \mathrm{Orb}_{G_n}$. We proceed to give a description of the latter, which only depends on G and not on the diagram of G_n's.

Definition 2.3.1 (Lie(G) subgroups)**.** Let G be a compactly metrizable group. We write $\mathrm{Sub}(G)$ for the poset of closed subgroups of G. For a closed subgroup $H \in \mathrm{Sub}(G)$ we write $H \in \mathrm{Lie}(G)$ if the G-action on the orbit G/H is inflated from an action by some Lie group. Phrased differently, we have $H \in \mathrm{Lie}(G)$ if and only if there exists a normal subgroup $N \trianglelefteq G$ with $N \le H$, so that the topological group G/N admits the structure of a Lie group.

Example 2.3.2. When G is profinite, then $\mathrm{Lie}(G)$ consists precisely of the open subgroups.

Lemma 2.3.3. Suppose morphisms $\varphi_n : G \to G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $G \cong \varprojlim_n G_n$ along surjective continuous homomorphism $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups. Let $H \le G$ be a closed subgroup of G. Then, $H \in \mathrm{Lie}(G)$ if and only if $\ker(\varphi_n) \le H$ for some $n \in \mathbb{N}$.

Proof. Suppose $N \le H$ is a normal subgroup of G with G/N a Lie group. By Theorem 2.1.4 applied to the quotient map $G \to G/N$, there exists $n \in \mathbb{N}$ with $\ker(\varphi_n) \le N \le H$. □

Definition 2.3.4 (Orbit Category)**.** Let G be a compactly metrizable group. The *orbit category* Orb_G is defined as the topological category with objects the orbit-spaces G/H for $H \in \mathrm{Lie}(G)$ and hom-spaces the set of G-equivariant maps between theses orbit spaces equipped with the compact-open topology. We use the same notation for Orb_G as topological category, as well as for the ∞-category Orb_G obtained as homotopy-coherent nerve.

We emphasize that Orb_G does not contain all quotients of closed subgroups of G, but only those whose canonical G-action arises via inflation from a compact Lie group.

In the situation of Equation (10), the fully faithful functors $\mathrm{Orb}_{G_n} \hookrightarrow \mathrm{Orb}_G$ assemble into an equivalence of ∞-categories

$$\operatorname{colim}_n \mathrm{Orb}_{G_n} \to \mathrm{Orb}_G.$$

Indeed, the essential surjectivity follows from Lemma 2.3.3. In conjunction with the previous discussion, we constructed a zig-zag of equivalences

$$\mathcal{S}_G \xleftarrow{\simeq} \mathcal{P}(\operatorname{colim}_n \mathrm{Orb}_{G_n}) \xrightarrow{\simeq} \mathcal{P}(\mathrm{Orb}_G) \tag{11}$$

from the ∞-category of *genuine G-spaces* to presheaves on the orbit category.

Construction 2.3.5 (Inflation as Kan-Extension)**.** For a surjective homomorphism $\varphi : K \to L$ of compactly metrizable groups with kernel $N \in \mathrm{Sub}(K)$, the composite $\mathrm{Orb}_L \to \mathcal{S}_L \xrightarrow{\mathrm{Infl}_L^K} \mathcal{S}_K$, factors through

$$\varphi^* : \mathrm{Orb}_L \to \mathrm{Orb}_K, \qquad L/U \mapsto K/\varphi^{-1}(U) \tag{12}$$

Indeed, unraveling the constructions, it suffices to check this for compact Lie groups. As φ^* is fully-faithful, its Yoneda extension

$$\mathrm{Infl}_L^K \simeq \mathrm{Lan}_{(\varphi^*)^{\mathrm{op}}} : \mathcal{S}_L \hookrightarrow \mathcal{S}_K$$

is fully faithful, too. Moreover, the functor φ^* admits an enriched left adjoint on the level of topological categories

$$(-)/N : \mathrm{Orb}_K \to \mathrm{Orb}_L, \qquad K/U \mapsto L/\varphi(U).$$

Yoneda extending yields the *quotient functor* $(-)/N : \mathcal{S}_K \to \mathcal{S}_L$, which is left adjoint to Infl_L^K. Moreover, under the description of G-spaces as category of presheaves, the fixed point functor $\varphi_* : \mathcal{S}_K \to \mathcal{S}_L$ identifies with precomposition by the opposite of φ^*.

Remark 2.3.6 (Large Orbits are Empty)**.** Suppose surjective homomorphisms $\varphi_n : G \to G_n$ exhibit a topological group as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit of compact Lie groups. For any $H \in \mathrm{Sub}(G)$, we may form the following inverse limit

$$(G/H)_{\mathrm{cont}} := \varprojlim \Big(\mathrm{Infl}_{G_0}^G(G_0/H_0) \leftarrow \mathrm{Infl}_{G_1}^G(G_1/H_1) \leftarrow \mathrm{Infl}_{G_2}^G(G_2/H_2) \leftarrow \cdots\Big) \in \mathcal{S}_G \tag{13}$$

with $H_n := \varphi_n(H) \le G_n$. Let $N_n := \ker(\varphi_n) \trianglelefteq G$. When $\mathrm{Map}_{\mathcal{S}_G}(G/N_n, (G/H)_{\mathrm{cont}}) \neq \emptyset$ for some $n \in \mathbb{N}$, then $N_n N_m \le H N_m$ for all $m \in \mathbb{N}$, so that $N_n \le H$, by [RZ00, Proposition 2.1.4.][2]. Thus, $(G/H)_{\mathrm{cont}} \neq \emptyset$ in $\mathcal{S}_G$ implies $H \in \mathrm{Lie}(G)$. In that case, the inverse limit $(G/H)_{\mathrm{cont}}$ stabilizes at $G/H \in \mathrm{Orb}_G$.

Construction 2.3.7 (Induction via Kan Extension)**.** Suppose surjective homomorphisms $\varphi_n : G \to G_n$ exhibit a topological group as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit of compact Lie groups G_n and let $H \le G$ be a closed subgroup. For any $U \in \mathrm{Lie}(H)$, $m \ge n \ge 0$ and $X \in \mathcal{S}_{G_n}$ one traces through the functorialities[3] to construct a preferred natural equivalence

$$\mathrm{Map}_{\mathcal{S}_G}\Big(\mathrm{Infl}_{G_m}^G(G_m/U_m), \mathrm{Infl}_{G_n}^G(X)\Big) \simeq \mathrm{Map}_{\mathcal{S}_H}\Big(H/U, \mathrm{Res}_H^G \mathrm{Infl}_{G_n}^G(X)\Big), \tag{14}$$

for $U_m = \varphi_m(U) \le H_m := \varphi_m(H) \le G_m$. Obverse that when $U \in \mathrm{Lie}(G)$, then $\mathrm{Infl}_{G_m}^G(G_m/U_m) = G/U$ for m big enough. Passing to colimits in the space X and varying $n \in \mathbb{N}$, we constructed for any $U \in \mathrm{Lie}(G) \cap \mathrm{Sub}(H)$ an equivalence

$$\mathrm{Map}_{\mathcal{S}_H}(H/U, \mathrm{Res}_H^G(X)) \simeq \mathrm{Map}_{\mathcal{S}_G}(G/U, X), \tag{15}$$

naturally in $X \in \mathcal{S}_G$. In the case that $H \in \mathrm{Lie}(G)$, then $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$, see [Fau08, Lem. 2.1.]. We conclude that the functor $\mathrm{Res}_H^G : \mathcal{S}_G \to \mathcal{S}_H$ identifies with the functor $\mathcal{P}(\mathrm{Orb}_G) \to \mathcal{P}(\mathrm{Orb}_H)$ given by precomposition with the topologically-enriched functor

$$G \times_H (-) : \ \mathrm{Orb}_H \to \mathrm{Orb}_G, \qquad H/U \mapsto G/U. \tag{16}$$

Via Yoneda-extending $G \times_H (-)$, we obtain a left adjoint of $\mathrm{Res}_H^G : \mathcal{S}_G \to \mathcal{S}_H$, denoted

$$\mathrm{Ind}_H^G : \mathcal{S}_H \to \mathcal{S}_G, \qquad H/U \mapsto G/U. \tag{17}$$

This left adjoint Ind_H^G is called *induction*. For its construction, we used that $H \in \mathrm{Lie}(G)$.

[2] Their proof of [RZ00, Proposition 2.1.4.] generalizes verbatim from profinite to compact Hausdorff groups.

[3] In detail, one may use the two different factorizations of the group homomorphism $H \to G_n$ and then the results discussed in Construction 2.3.5.

Remark 2.3.8 (No Induction Possible)**.** Suppose surjective morphisms $\varphi_n : G \to G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $\varprojlim_n G_n$ of compact Lie groups G_n. Let us write $N_n := \ker(\varphi_n)$ for the kernel of φ_n. Let $H \in \mathrm{Sub}(G)$ be a closed subgroup. Let us assume that $\mathrm{Res}^G_H : \mathcal{S}_G \to \mathcal{S}_H$ admits a left adjoint Ind^G_H. As $\mathrm{Res}^G_H(-)$ preserves filtered colimits, its left adjoint preserves compact objects. In particular, Ind^G_H sends the terminal object $* \in \mathcal{S}_H$ to some

$$\mathrm{Ind}^G_H(*) = \mathrm{Infl}^G_{G_n}(X) \in \mathcal{S}_G$$

for some $X \in \mathcal{S}^\omega_{G_n}$ and $n \in \mathbb{N}$. Unraveling the definition and applying Equation (14), for any $m \geq n$, we obtain an equivalence

$$\begin{aligned}\mathrm{Map}_{\mathcal{S}_{G_m}}(\mathrm{Infl}^{G_m}_{G_n} X, Y) &\simeq \mathrm{Map}_{\mathcal{S}_G}\left(\mathrm{Infl}^G_{G_n}(X), \mathrm{Infl}^G_{G_m}(Y)\right) \simeq \mathrm{Map}_{\mathcal{S}_H}\left(H/H, \mathrm{Res}^G_H \mathrm{Infl}^G_{G_m}(Y)\right) \\ &\overset{(14)}{\simeq} \mathrm{Map}_{\mathcal{S}_G}\left(\mathrm{Infl}^G_{G_m}(G_m/H_m), \mathrm{Infl}^G_{G_m}(Y)\right) \simeq \mathrm{Map}_{\mathcal{S}_{G_m}}(G_m/H_m, Y)\end{aligned}$$

naturally in $Y \in \mathcal{S}_{G_m}$, where $H_m := \varphi_m(H) \leq G_m$. By the Yoneda Lemma, $\mathrm{Infl}^G_{G_n}(X) = \mathrm{Infl}^G_{G_m}(G_m/H_m) = G/(HN_m)$. As this was independent of the choice of $m \geq n$, we have $HN_n = HN_m$ for all $m \geq n$. As in Remark 2.3.6, we conclude $N_n \leq H$, so that $H \in \mathrm{Lie}(G)$.

Lemma 2.3.9. Let G be a compactly metrizable group and $H \in \mathrm{Lie}(G)$ a subgroup. Let $\varphi_n : G \to G_n$ be a surjective continuous group homomorphism and set $H_n := \varphi_n(H) \leq G_n$. If $\ker(\varphi_n) \leq H$, then the Beck-Chevalley transformation of the natural equivalence

$$\mathrm{Infl}^G_{H_n} \circ \mathrm{Res}^{G_n}_{H_n} \overset{\simeq}{\Longrightarrow} \mathrm{Res}^G_H \circ \mathrm{Infl}^G_{G_n}$$

is an equivalence

$$\mathrm{Ind}^G_H \circ \mathrm{Infl}^H_{H_n} \overset{\simeq}{\Longrightarrow} \mathrm{Infl}^G_{G_n} \mathrm{Ind}^{G_n}_{H_n}$$

of functors $\mathcal{S}_{H_n} \to \mathcal{S}_G$.

Proof. Because all of the above functors preserve colimits, it suffices to check that the Beck-Chevalley transformation is an equivalence on oribits $H_n/K \in \mathcal{S}_{H_n}$ for $K \in \mathrm{Lie}(H_n)$. By the description of induction and inflation as left Kan extensions, see (2.3.7) and (2.3.5), the Beck-Chevalley transformation evaluates at $H_n/K \in \mathrm{Orb}_{H_n}$ to the canonical map

$$G/(\varphi_n^{-1}(K) \cap H) \to G/\varphi_n^{-1}(K)$$

in Orb_G. By assumption, we have $\varphi_n^{-1}(K) \leq \varphi_n^{-1}(H_n) = H$, so that $\varphi_n^{-1}(K) \cap H = \varphi_n^{-1}(K)$. ☐

Definition 2.3.10 (Fixed Points)**.** We write $(-)^G : \mathcal{S}_G \to \mathcal{S}$ for the right-adjoint of $\mathrm{Infl}^G_1(-)$. More generally for, $H \in \mathrm{Sub}(G)$, we call the composite

$$(-)^H : \mathcal{S}_G \xrightarrow{\mathrm{Res}^G_H} \mathcal{S}_H \xrightarrow{(-)^H} \mathcal{S} \tag{18}$$

H-fixed points.

Whenever $H \in \mathrm{Lie}(G)$, then the H-fixed point functor $(-)^H$ is co-represented by the orbit $G/H \in \mathrm{Orb}_G$.

Lemma 2.3.11. Let $\varphi : G \to G_n$ be a surjective continuous homomorphism of compactly metrizable groups. Let $H \leq G$ be a closed subgroup and $H_n := \varphi(H)$ its image. Then, we obtain a preferred equivalence

$$\left(\mathrm{Infl}^G_{G_n}(-)\right)^H \simeq (-)^{H_n} \tag{19}$$

of functors $\mathcal{S}_{G_n} \to \mathcal{S}$.

Proof. Unraveling Construction 2.3.5 we see that $(-)^H \circ \mathrm{Infl}_{H_n}^H \simeq (-)^{H_n}$, so that

$$\left(\mathrm{Infl}_{G_n}^G(-)\right)^H = (-)^H \circ \mathrm{Res}_H^G \circ \mathrm{Infl}_{G_n}^G \simeq (-)^H \circ \mathrm{Infl}_{H_n}^H \circ \mathrm{Res}_{H_n}^{G_n} \simeq (-)^{H_n} \circ \mathrm{Res}_{H_n}^{G_n} = (-)^{H_n},$$

where we factored the group homomorphism $H \to G_n$ in two different ways. □

2.3.1. *Based Genuine G-Spaces.*

Construction 2.3.12. The functor $\mathcal{S}_\bullet : \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \to \mathrm{Pr}^L$ from Construction 2.2.3 admits a lift along the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^L) \to \mathrm{Pr}^L$, where for any $G \in \mathrm{Pro}(\mathrm{CptLie})$ the symmetric monoidal structure on $\mathcal{S}_G$ is cartesian. For $G \in \mathrm{Pro}(\mathrm{CptLie})$, we define the category of pointed G-spaces $\mathcal{S}_{G,*}$ as the value of the composite

$$\mathcal{S}_{\bullet,*} : \ \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \xrightarrow{\mathcal{S}_\bullet} \mathrm{CAlg}(\mathrm{Pr}^L) \xrightarrow{\mathcal{S}_* \otimes (-)} \mathrm{CAlg}(\mathrm{Pr}^L)$$

at G. The underlying ∞-category of $\mathcal{S}_{G,*} \simeq (\mathcal{S}_G)_{*/}$ canonically identifies with the category of pointed objects in $\mathcal{S}_G$ by [Lur17, Example 4.8.1.21].

Notation 2.3.13. For a compactly-metrizable topological group G we fix the following notation:

- The tensor product of pointed G-spaces is denoted by $(-) \wedge (-)$.
- Following Convention 2.2.6, we analogously define restriction and inflation, in the based setting and use the same notation for their right adjoints.
- Naturally in $G \in \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}}$, we have a symmetric monoidal functor

$$(-)_+ : \mathcal{S}_G \to \mathcal{S}_{G,*}$$

whose right adjoint is the *forgetful functor* $\mathrm{fgt} : \mathcal{S}_{G,*} \to \mathcal{S}_G$ from the slice.

The properties of the slice category imply the following facts:

(1) The forgetful functor $\mathrm{fgt} : \mathcal{S}_{G,*} \to \mathcal{S}_G$ is conservative, detects compact objects, and preserves weakly contractible colimits.

(2) For a continuous homomorphism $\varphi : G \to K$ of compactly-metrizable groups, the induced symmetric monoidal functor $\varphi^* : \mathcal{S}_{K,*} \to \mathcal{S}_{G,*}$ satisfies

$$\mathrm{fgt} \circ \varphi^* \ \simeq \ \varphi^* \circ \mathrm{fgt}$$

because $\varphi^* : \mathcal{S}_K \to \mathcal{S}_G$ preserves the point.

(3) For $H \in \mathrm{Sub}(G)$ the *fixed point functor* $(-)^H : \mathcal{S}_{G,*} \to \mathcal{S}$ is defined as the composite

$$\mathcal{S}_{G,*} \xrightarrow{\mathrm{fgt}} \mathcal{S}_G \xrightarrow{(-)^H} \mathcal{S}\ .$$

The fixed point functor preserves weakly contractible colimits. When, $H \in \mathrm{Lie}(G)$, then $(-)^H : \mathcal{S}_{G,*} \to \mathcal{S}$ is co-represented by G/H_+. The objects $\{G/H_+\}_{H \in \mathrm{Lie}(G)}$ are jointly conservative and generate $\mathcal{S}_{G,*}$ under small colimits.

(4) For $H \in \mathrm{Sub}(G)$ the fixed point functor $(-)^H : \mathcal{S}_{G,*} \to \mathcal{S}$ has a unique lift along the forgetful functor $\mathcal{S}_* \to \mathcal{S}$, given by the composite

$$(-)^H : \ \mathcal{S}_{G,*} \xrightarrow{\mathrm{Res}_H^G} \mathcal{S}_{H,*} \xrightarrow{(\varphi_0)_*} \mathcal{S}_* \tag{20}$$

for $(\varphi_0)_*$ right adjoint to $\mathrm{Infl}_1^H : \mathcal{S}_* \to \mathcal{S}_H$.

2.4. **Classifying Spaces of Families.** The following discussion closely parallels the compact Lie group equivariant case.

Definition 2.4.1. A *family* $\mathcal{F}$ *of subgroups* of a compactly metrizable group G is a subset $\mathcal{F} \subseteq \mathrm{Lie}(G)$, which is closed under conjugation and subgroups. We write $\mathrm{Orb}_G(\mathcal{F}) \subseteq \mathrm{Orb}_G$ for the full subcategory spanned by $\{G/U \colon U \in \mathcal{F}\}$. We write $\mathcal{S}_G^{\mathcal{F}} := \mathcal{P}(\mathrm{Orb}_G(\mathcal{F}))$ for presheaves on $\mathrm{Orb}_G(\mathcal{F})$.

Example 2.4.2. If $N \leq G$ is the kernel of a surjective morphism $\varphi : G \to K$ of compactly metrizable topological groups, the set $\mathcal{P}_N := \{U \in \mathrm{Lie}(G) : N \nleq U\}$ is a family of subgroups.

For the rest of Section 2.4 we fix a family $\mathcal{F}$ of subgroups of a compactly metrizable group G.

Construction 2.4.3. Yoneda-extending $\mathrm{Orb}_G(\mathcal{F}) \to \mathrm{Orb}_G$ yields a fully faithful functor

$$i : \mathcal{S}_G^{\mathcal{F}} \hookrightarrow \mathcal{S}_G \qquad \text{with colimit preserving right adjoint} \qquad \Psi : \mathcal{S}_G \to \mathcal{S}_G^{\mathcal{F}} .$$

By construction, we have $(i \circ \Psi(X))^U = X^U$ for all $U \in \mathcal{F}$ and $X \in \mathcal{S}_G$.

Definition 2.4.4. A G-space $X \in \mathcal{S}_G$ is called $\mathcal{F}$*-torsion* if X is in the essential image of $i : \mathcal{S}_G^{\mathcal{F}} \hookrightarrow \mathcal{S}_G$. The full subcategory spanned by $\mathcal{F}$-torsion spaces is denoted $(\mathcal{S}_G)_{\mathcal{F}\text{-tors}} \subseteq \mathcal{S}_G$.

The subcategory $(\mathcal{S}_G)_{\mathcal{F}\text{-tors}}$ is the smallest full subcategory of $\mathcal{S}_G$, which contains $\mathrm{Orb}_G(\mathcal{F})$ and is closed under colimits.

Lemma 2.4.5. A G-space $X \in \mathcal{S}_G$ is $\mathcal{F}$-torsion if and only if $X^U = \emptyset$ for all $U \in \mathrm{Lie}(G) \setminus \mathcal{F}$.

Proof. If $U \in \mathrm{Lie}(G)$ and $W \in \mathcal{F}$, such that there exists $gW \in (G/W)^U$, then $g^{-1}Ug \leq W$ implies $U \in \mathcal{F}$. Thus, the pointwise formula for left Kan extension implies that $(i\Psi(X))^U$ is a colimit of constant diagram at the initial object $\emptyset$, whenever $U \notin \mathcal{F}$. □

Definition 2.4.6. We define the *classifying space* $E\mathcal{F} \in \mathcal{S}_G$ *of the family* $\mathcal{F}$ as the terminal object of $(\mathcal{S}_G)_{\mathcal{F}-\mathrm{tors}}$.

The classifying space $E\mathcal{F} \in \mathcal{S}_G$ is uniquely determined by its fixed points

$$(E\mathcal{F})^U = \begin{cases} *, & U \in \mathcal{F} \\ \emptyset, & U \notin \mathcal{F} . \end{cases}$$

Remark 2.4.7. The endofunctor $\Psi \circ i$ is equivalent to $E\mathcal{F} \times (-)$. Under this equivalence, the co-unit at $X \in \mathcal{S}_G$ identifies with $E\mathcal{F} \times X \xrightarrow{\mathrm{pr}} X$.

Remark 2.4.8. The functor $\mathcal{S}_G^{\mathcal{F}} = \left(\mathcal{S}_G^{\mathcal{F}}\right)_{/*} \xrightarrow{i} (\mathcal{S}_G)_{/E\mathcal{F}}$ is an equivalence of categories.

Construction 2.4.9. We view $\mathcal{S}_G^{\mathcal{F}}$ as cartesian monoidal and endow $\mathcal{S}_{G,*}^{\mathcal{F}} \simeq \mathcal{S}_G^{\mathcal{F}} \otimes^L \mathcal{S}_*$ with the tensor-product algebra structure in Pr^L. The functor $\Psi : \mathcal{S}_G \to \mathcal{S}_G^{\mathcal{F}}$ induces a symmetric monoidal left-adjoint functor

$$\Psi : \mathcal{S}_{G,*} \to \mathcal{S}_{G,*}^{\mathcal{F}}$$

which commutes with forgetting the base-point. By [Lur09, 5.2.5.1.], Ψ admits a left adjoint, denoted i_*, that sends $(* \xrightarrow{f} X) \in \mathcal{S}_{G,*}^{\mathcal{F}}$ to the cofiber of $i(f) : E\mathcal{F} \to i(X)$. In particular, the endo-functor

$$i_* \circ \Psi : \mathcal{S}_{G,*} \to \mathcal{S}_{G,*}, \qquad \text{is equivalent to} \qquad X \mapsto E\mathcal{F}_+ \wedge X.$$

Under this equivalence, the co-unit identifies with the projection $E\mathcal{F}_+ \wedge X \to X$.

As $i : \mathcal{S}_G^{\mathcal{F}} \to \mathcal{S}_G$ preserves products, we conclude

Lemma 2.4.10. The functor $i_* : \mathcal{S}^{\mathcal{F}}_{G,*} \to \mathcal{S}_{G,*}$ is symmetric monoidal and fully-faithful. For $X \in \mathcal{S}_{G,*}$, the co-unit induces an equivalence $(i_* \circ \Psi(X))^U \to X^U$ for all $U \in \mathcal{F}$. The co-unit $i_* \circ \Psi(X) \to X$ is an equivalence if and only if $X^U = *$ for all $U \notin \mathcal{F}$.

Definition 2.4.11. We say that $X \in \mathcal{S}_{G,*}$ is $\mathcal{F}^{-1}$*-local* if $E\,\mathcal{F}_+ \wedge X$ is a terminal object. Equivalently, we may demand that $X^U \simeq *$ for all $U \in \mathcal{F}$. We write $\mathcal{S}_{G,*}[\mathcal{F}^{-1}] \subseteq \mathcal{S}_{G,*}$ for the full-subcategory spanned by $\mathcal{F}^{-1}$-local objects.

Observation 2.4.12. The category $\mathcal{S}_{G,*}[\mathcal{F}^{-1}] \subseteq \mathcal{S}_{G,*}$ is closed under both limits and colimits.

Definition 2.4.13. We define $\widetilde{E\,\mathcal{F}} \in \mathcal{S}_{G,*}$ via the cofiber sequence

$$E\,\mathcal{F}_+ \longrightarrow S^0 \xrightarrow{e} \widetilde{E\,\mathcal{F}}$$

of pointed G-spaces. From the definitions we deduce the following Lemma 2.4.14.

Lemma 2.4.14. For $X \in \mathcal{S}_{G,*}$ the morphism $X \xrightarrow{e\wedge 1} \widetilde{E\,\mathcal{F}} \wedge X$ serves as a unit for a left adjoint of the inclusion $\mathcal{S}_{G,*}[\mathcal{F}^{-1}] \subseteq \mathcal{S}_{G,*}$. This unit induces an equivalence on U-fixed points for all $U \notin \mathcal{F}$.

In particular, $S^0 \xrightarrow{e} \widetilde{E\,\mathcal{F}}$ exhibits $\widetilde{E\,\mathcal{F}}$ as an idempotent algebra in $\mathcal{S}_{G,*}$.

Example 2.4.15. In the situation of Example 2.4.2, the N*-fixed-point functor* $\varphi_* : \mathcal{S}_{G,*} \to \mathcal{S}_{K,*}$, i.e. the right adjoint to inflation Infl_K^G, induces an equivalence

$$\varphi_*(X) \xrightarrow{e\wedge 1} \varphi_*(\widetilde{E\,\mathcal{P}_N} \wedge X)$$

for all $X \in \mathcal{S}_{G,*}$. Indeed, this can be checked on fixed points and, similarly, the counit induces an equivalence

$$\widetilde{E\,\mathcal{P}_N} \wedge \mathrm{Infl}_K^G(\varphi_*(X)) \xrightarrow{\simeq} \widetilde{E\,\mathcal{P}_N} \wedge X.$$

We deduce that the composite

$$\mathcal{S}_{K,*} \xrightarrow{\mathrm{Infl}_K^G(-)} \mathcal{S}_{G,*} \xrightarrow{\widetilde{E\,\mathcal{P}_N}\wedge(-)} \mathcal{S}_{G,*}$$

defines a fully faithful right adjoint of the N-fixed point functor $\varphi_* : \mathcal{S}_{G,*} \to \mathcal{S}_{K,*}$. A pointed G-space $X \in \mathcal{S}_{G,*}$ is in the essential image of the right adjoint of φ_* if and only if X is $\mathcal{P}_N^{-1}$-local.

2.4.1. *Families for Pro Compact Lie Groups.* We assume, that a set of surjective group homomorphisms $\varphi_n : G \twoheadrightarrow G_n$ exhibit a topological group G as the following inverse limit

$$\varprojlim \left(G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_3} G_3 \xleftarrow{\phi_4} \cdots \right) \tag{21}$$

in the category of topological groups, with each $G_n \in \mathrm{CptLie}$ a compact Lie group.

Definition 2.4.16. For any family of subgroups $\mathcal{F} \subseteq \mathrm{Lie}(G)$, in the sense of Definition 2.4.1, we call the family

$$\mathcal{F}_n := \{U \leq G_n : \varphi_n^{-1}(U) \in \mathcal{F}\} \subseteq \mathrm{Sub}(G_n)$$

of subgroups of G_n the n^{th}*-stage* of $\mathcal{F}$.

Example 2.4.17. If $N \leq G$ is a closed normal subgroup of G, and $\mathcal{F} = \mathcal{P}_N$ is the family from Example 2.4.2, then the two families $\mathcal{F}_n = \mathcal{P}_{N_n}$ coincide for $N_n := \varphi_n(N)$. In particular, if $\mathcal{F}$ is the family of $H \in \mathrm{Lie}(G)$ with $H \neq G$, then $\mathcal{F}_n \subseteq \mathrm{Sub}(G_n)$ is the family of proper subgroups.

Because φ_n is surjective, the set $\mathcal{F}_n \subseteq \mathrm{Sub}(G_n)$ is a family of subgroups of G_n, in the sense of Definition 2.4.1. Moreover, as full subcategories of the orbit category Orb_G, we have

$$\mathrm{Orb}_G(\mathcal{F}) \cap \mathrm{Infl}_{G_n}^G(\mathrm{Orb}_{G_n}) = \mathrm{Infl}_{G_n}^G(\mathrm{Orb}_{G_n}(\mathcal{F}_n)) = \{G/\varphi_n^{-1}(U) : U \in \mathcal{F}_n\}.$$

By definition of the family $\mathrm{Lie}(G) \subseteq \mathrm{Sub}(G)$, we see that the inflations $\mathrm{Infl}_{G_n}^G : \mathrm{Orb}_{G_n}(\mathcal{F}_n) \to \mathrm{Orb}_G(\mathcal{F})$ exhibit the ∞-category $\mathrm{Orb}_G(\mathcal{F})$ as the colimit of the diagram

$$\mathrm{Orb}_{G_0}(\mathcal{F}_0) \hookrightarrow \mathrm{Orb}_{G_1}(\mathcal{F}_1) \hookrightarrow \mathrm{Orb}_{G_2}(\mathcal{F}_2) \hookrightarrow \mathrm{Orb}_{G_3}(\mathcal{F}_3) \hookrightarrow \cdots$$

in Cat_∞. We conclude that the unique map of G-spaces

$$\mathrm{colim}_n\, \mathrm{Infl}_{G_n}^G(E\mathcal{F}_n) \to E\mathcal{F} \tag{22}$$

is an equivalence. Moreover, the inflations $\mathrm{Infl}_{G_n}^G : (\mathcal{S}_{G_n})_{\mathcal{F}_n\text{-tors}} \to (\mathcal{S}_G)_{\mathcal{F}\text{-tors}}$ exhibit the full subcategory $(\mathcal{S}_G)_{\mathcal{F}\text{-tors}} \subseteq \mathcal{S}_G$ of $\mathcal{F}$-tors spaces as the colimit of the diagram

$$\mathrm{colim}^L\left((\mathcal{S}_{G_0})_{\mathcal{F}_0\text{-tors}} \hookrightarrow (\mathcal{S}_{G_1})_{\mathcal{F}_1\text{-tors}} \hookrightarrow (\mathcal{S}_{G_2})_{\mathcal{F}_2\text{-tors}} \hookrightarrow \cdots\right) \xrightarrow{\simeq} (\mathcal{S}_G)_{\mathcal{F}\text{-tors}} \tag{23}$$

in the category of presentable ∞-categories and left adjoint functors Pr^L.

We defined the pointed G-space $\widetilde{E\mathcal{F}}$ as the cofiber of $(E\mathcal{F})_+ \to S^0$. The equivalence (22) induces an equivalence of pointed G-spaces

$$\mathrm{colim}_n\, \mathrm{Infl}_{G_n}^G\left(\widetilde{E\mathcal{F}_n}\right) \xrightarrow{\simeq} \widetilde{E\mathcal{F}} \tag{24}$$

under $S_0 \in \mathcal{S}_{G,*}$.

2.4.2. *The Model Structure on Topological G-Spaces.* Before we conclude our discussion of unstable equivariant homotopy theory, we would like to recall the connection to topological spaces with a group action. For this purpose, consider a compactly metrizable topological group G and $\mathcal{F} \subseteq \mathrm{Lie}(G)$ a family of subgroups. We let Top_G denote the (topological) category of compactly generated weak Hausdorff topological G-spaces.

Proposition 2.4.18 ([Sch18, Proposition B.7.]). *A G-map $f \in \mathrm{Top}_G(X,Y)$ is called an $\mathcal{F}$-equivalence ($\mathcal{F}$-fibration) if for all $H \in \mathcal{F}$ the map induced on fixed points $f^H : X^H \to Y^H$ is a weak-equivalence (Serre fibration) of topological spaces.*

(i) The $\mathcal{F}$-equivalences and $\mathcal{F}$-fibrations form the weak equivalences and fibrations of a proper, topological, cellular model structure on the category Top_G *of G-spaces,* the $\mathcal{F}$-projective model structure*, where all objects of* Top_G *are $\mathcal{F}$-fibrant.*

(ii) The set of maps

$$I_\mathcal{F} := \{G/H \times i^k\colon\ G/H \times \partial D^k \to G/H \times D^k\}_{k\geq 0, H\in\mathcal{F}} \tag{25}$$

serves as a set of generating cofibrations for the $\mathcal{F}$-projective model structure. The set of maps

$$J_\mathcal{F} = \{G/H \times j^k\colon\ G/H \times D^k \times \{0\} \to G/H \times D^k \times [0,1]\}_{k\geq 0, H\in\mathcal{F}}$$

serves as a set of generating acyclic cofibrations.

(iii) Equivariant Whitehead Theorem: Two maps are left homotopic in the $\mathcal{F}$-projective model structure on Top_G *if and only if there is a homotopy $H : f \simeq g$ in* Top *such that H_t is G-equivariant for all $t \in [0,1]$. In particular, a G-map $f : X \to Y$ between $\mathcal{F}$-cofibrant G-spaces is a $\mathcal{F}$-equivalence if and only if f has a homotopy inverse established via G-equivariant homotopies.*

Proposition 2.4.19 (Elmendorf's Theorem, [LNP25, Theorem 3.39.])**.** *Yoneda extending the inclusion* $\mathrm{Orb}_G(\mathcal{F}) \to \mathrm{Top}_G$ *induces an equivalence*

$$\mathcal{S}_G^{\mathcal{F}} := \mathcal{P}(\mathrm{Orb}_G(\mathcal{F})) \xrightarrow{\simeq} (\mathrm{Top}_G)^{\mathcal{F},\circ}$$

of ∞*-categories. Here,* $(\mathrm{Top}_G)^{\mathcal{F},\circ}$ *denotes (the homotopy-coherent nerve of) the topological subcategory of* Top_G *spanned by* $\mathcal{F}$*-cofibrant* G*-spaces.*

Proof. The following fact can be extracted from [Sch18, Appendix B.]: The orbits G/H for $H \in \mathcal{F}$ form a jointly conservative set of tiny objects[4] in the ∞-category $(\mathrm{Top}_G)^{\mathcal{F},\circ}$. It is a formal consequence from this, that the above functor out of $\mathcal{P}(\mathrm{Orb}_G(\mathcal{F}))$ is an equivalence of ∞-categories, see [LNP25, Theorem 3.39.]. □

Remark 2.4.20. Consider the localization $\mathrm{Top}_G[\mathcal{F}\,\text{−equiv.}^{-1}]$, in the ∞-categorical sense, of the 1-category of $\mathcal{F}$-cofibrant G-spaces by the $\mathcal{F}$-equivalences. Dwyer and Kan, see [Lur17, Theorem 1.3.4.20.], established that the canonical functor

$$\mathrm{Top}_G[\mathcal{F}\,\text{−equiv.}^{-1}] \xrightarrow{\simeq} (\mathrm{Top}_G)^{\mathcal{F},\circ}$$

is an equivalence of ∞-categories.

A continuous homomorphism $\alpha\colon K \to G$ between compactly metrizable groups gives rise to Top-enriched adjoint functors between the associated category of equivariant spaces

$$\mathrm{Top}_G \begin{array}{c} \xleftarrow{G\times_\alpha(-)} \\ \xrightarrow{\alpha^*} \\ \xleftarrow[\underline{\mathrm{Hom}}^\alpha(G,-)]{} \end{array} \mathrm{Top}_K$$

and [Fau08, Lemma 2.2.] reads:

Proposition 2.4.21. *Let* $\mathcal{F} \subseteq \mathrm{Lie}(K)$ *and* $\mathcal{G} \subseteq \mathrm{Lie}(G)$ *be families of closed subgroups and equip* Top_K *and* Top_G *with the* $\mathcal{F}$*-projective (respectively,* $\mathcal{G}$*-projective) model structure.*

- *(a) If* $\alpha(\mathcal{F}) \subseteq \mathcal{G}$*, then* α^* *preserves fibrations and weak-equivalences. In particular, the adjoint pair* $(G \times_\alpha -, \alpha^*)$ *is Quillen.*
- *(b) If* $\alpha^{-1}(\mathcal{G}) \subseteq \mathcal{F}$*, then the adjoint pair* $(\alpha^*, \underline{\mathrm{Hom}}^\alpha(G,-))$ *is also Quillen.*

Remark 2.4.22. If we are in the special situation that $\mathcal{F} = \mathrm{Lie}(K)$ and $\mathcal{G} = \mathrm{Lie}(G)$, then $\alpha^{-1}(\mathcal{G}) \subseteq \mathcal{F}$ is automatic and $\alpha(\mathcal{F}) \subseteq \mathcal{G}$ holds if and only if $\mathrm{Im}(\alpha) \in \mathrm{Lie}(G)$.

Under Elmendorf's theorem, the above Quillen functors agree with our previous constructions:

- (a) If $\mathrm{Im}(\alpha) \in \mathrm{Lie}(G)$, then the left adjoint of the functor of ∞-categories $\alpha^* : \mathcal{S}_G \to \mathcal{S}_K$ from Convention 2.2.6 is given by left Kan extension along $G \times_\alpha (-) : \mathrm{Orb}_K \to \mathrm{Orb}_G$, see 2.3.5 and 2.3.7. If, additionally, $\alpha(\mathcal{F}) \subseteq \mathcal{G}$, then, the left Kan extension restricts to a left adjoint functor $\mathcal{S}_K^{\mathcal{F}} \to \mathcal{S}_G^{\mathcal{G}}$. This functor is canonically equivalent to the left adjoint functor induced from the Quillen adjunction of Proposition 2.4.21 a). Indeed, both of these functors are left Kan extended from the same functor $\mathrm{Orb}_K(\mathcal{F}) \to \mathrm{Orb}_G(\mathcal{G})$.
- (b) If $\alpha^{-1}(\mathcal{F}) \subseteq \mathcal{G}$, then the functor of ∞-categories $\alpha^* : \mathcal{S}_G \to \mathcal{S}_K$ from Construction 2.2.3 restricts to a functor $\mathcal{S}_G^{\mathcal{G}} \to \mathcal{S}_K^{\mathcal{F}}$ which is canonically equivalent to the functor induced from the Quillen adjunction of Proposition 2.4.21 (b). Indeed, these functors agree, when restricted to $\mathrm{Orb}_G(\mathcal{F})$.

2.5. **Equivariant Principal Bundles and Universes.**

[4] An object of an ∞-category $\mathcal{C}$ is called *tiny* if the functor $\mathcal{C} \to \mathcal{S}$ co-represented by that object preserves small colimits.

2.5.1. *Equivariant Principal Bundles.* We recollect the conclusions from [LU14] and translate them into our language. Most results go back at least to [Las82, Lashof]. In this Section 2.5.1, we fix both a compact Lie group L and a compactly metrizable group G.

Definition 2.5.1. A *G-equivariant L-principal bundle* is a morphism $p : E \to B$ in $\mathrm{Top}_{G\times L}$ s.t.

(1) $B \cong \mathrm{Infl}_G^{G\times L}(X)$ for some $\mathrm{Lie}(G)$-cofibrant $X \in \mathrm{Top}_G$, and,
(2) the L-equivariant map $\mathrm{Res}_L^{G\times L}(p) : E \to B$ is an L-principal bundle.

The G-equivariant L-principal bundles span a full topological subcategory

$$\mathrm{Bun}_{(G,L)}^{\mathrm{Top}} \subseteq \mathrm{Fun}([1], \mathrm{Top}_{G\times L}).$$

We write $\mathrm{Bun}_{(G,L)} := N^{\mathrm{hc}}\left(\mathrm{Bun}_{(G,L)}^{\mathrm{Top}}\right)$ for the ∞-category of G-equivariant L-principal bundles.

Definition 2.5.2 (Graph Subgroups)**.** For a compact Lie group L, a compactly metrizable group G, a subgroup $H \in \mathrm{Lie}(G)$ and a continuous group homomorphism $\alpha : H \to L$, we write

$$\Gamma(\alpha) := \{(h, \alpha(h)) : h \in H\} \leq G \times L$$

for its *graph subgroup.*

By Theorem 2.1.4, $\ker(\alpha) \in \mathrm{Lie}(G)$, so that $\Gamma(\alpha) \in \mathrm{Lie}(G \times L)$. The collection of all graph subgroups form a family $\mathcal{F}_\Gamma$ of subgroups of $G \times L$.

Proposition 2.5.3. *Let $p : E \to B$ be a G-equivariant L-principal bundle, then $E \in \mathrm{Top}_{G\times L}$ is $\mathcal{F}_\Gamma$-cofibrant.*

Proof. We call a $\mathrm{Lie}(G)$-cofibrant space B *good* if the statement holds for all G-equivariant L-principal bundles over B. If $B = \mathrm{Infl}_{G_n}^G(G_n/H_n) \times Z$ for G_n a compact Lie group, $H_n \in \mathrm{Lie}(G_n)$ and $Z \in \mathrm{Top}$ a cofibrant non-equivariant space, then by [LU14, Lemma 6.2.], there exists a continuous homomorphism $\alpha : H_n \to L$ and a $(G_n \times L)$-homeomorphism $(G_n \times_\alpha L) \times Z \to E$ over the quotient map to B. We conclude that any pushout of base spaces along a generating cofibration[5] $f \in I_{\mathrm{Lie}(G)}$ pulls back to a pushout of total spaces along a generating cofibration $f' \in I_{\mathcal{F}_\Gamma}$. In particular, the class of good B is closed under pushouts along the generating $\mathrm{Lie}(G)$-cofibrations $f \in I_{\mathrm{Lie}(G)}$. Similarly, we deduce that a filtered colimit along pushouts of morphisms in $I_{\mathrm{Lie}(G)}$ on base spaces pulls back to a filtered colimit along pushouts of morphisms in $I_{\mathcal{F}_\Gamma}$ on total spaces. It remains to be proven that the class of good B is closed under retracts. We may pull back a bundle $p : E \to B$ along a G-map $B' \xrightarrow{r} B$, to exhibit the total space E as a retract of a $\mathcal{F}^\Gamma$-cofibrant space, whenever B' is good and r admits a section. □

Proposition 2.5.4. *For any $E \in \mathrm{Top}_{G\times L}$ which is $\mathcal{F}_\Gamma$-cofibrant, the counit $p : E \to E/L$ is a G-equivariant L-principal bundle.*

Proof. By Proposition 2.4.21, it remains to be shown that $\mathrm{Res}_L(p)$ is an L-principle bundle. We first treat the case that $E = [(G \times L)/\Gamma(\alpha)] \times Z$ for some non-equivariant cofibrant $Z \in \mathrm{Top}$, where $\alpha : H \to L$ is a continuous group-homomorphism for $H \in \mathrm{Lie}(G)$. We chose some normal subgroup $N \trianglelefteq G$ so that $G_n := G/N$ is a compact Lie-group with $N \leq H$. Then, $G/H = G_n/H_n$, where $H_n := H/N \leq G_n$, and E is inflated from the $(G_n \times L)$-space $E' := (G_n \times_\alpha L) \times Z$. The quotient map $E' \to E'/L$ is an L-principle bundle by Palais slice theorem, see [LU14, Lemma 5.1.]. We conclude that the class of *good* $E \in \mathrm{Top}_{G\times L}$, i.e. those for which the conclusion holds, is closed under pushouts along the generating $\mathcal{F}_\Gamma$-cofibrations $I_{\mathcal{F}_\Gamma}$ from Equation (25). The class of good $E \in \mathrm{Top}_{G\times L}$ is also closed under filtered colimits along pushouts along the generating $\mathcal{F}_\Gamma$-cofibrations: Indeed, L-principal bundles are stable under filtered colimits along

[5] See Equation (25) for the definition of $I_{\mathrm{Lie}(G)}$.

cofibrations of underlying topological spaces. Lastly, consider $(G \times L)$-maps $E \xrightarrow{s} E' \xrightarrow{r} E$ with good $\mathcal{F}$-cofibrant E' and $rs = \mathrm{id}$. Then, a local trivialization of $E' \to E'/L$ restricts along s to a local trivialization of $E \to E/L$. □

Construction 2.5.5. We obtain topologically enriched functors

$$(\mathrm{Top}_{G\times L})^{\mathcal{F}_\Gamma,\circ} \to \mathrm{Bun}^{\mathrm{Top}}_{(G,L)}, \qquad E \mapsto (E \to E/L) \qquad \text{and}$$
$$\mathrm{Bun}^{\mathrm{Top}}_{(G,L)} \to (\mathrm{Top}_{G\times L})^{\mathcal{F}_\Gamma,\circ}, \qquad (E \xrightarrow{p} B) \mapsto E,$$

where the composite $(\mathrm{Top}_{G\times L})^{\mathcal{F}_\Gamma,\circ} \to (\mathrm{Top}_{G\times L})^{\mathcal{F}_\Gamma,\circ}$ is equal to the identity functor. The quotient-inflation adjunction 2.4.21 provides us with a homeomorphism $E/L \to B$ under E, naturally in the G-equivariant L-principal fibration $p : E \to B$.

Corollary 2.5.6. *The functors from Construction 2.5.5 are mutually inverse topologically enriched functors, establishing an equivalence of topological categories*

$$\mathrm{Bun}^{\mathrm{Top}}_{(G,L)} \cong (\mathrm{Top}_{G\times L})^{\mathcal{F}_\Gamma,\circ}, \qquad \textit{which induces an equivalence} \qquad \mathrm{Bun}_{(G,L)} \simeq \mathcal{S}^{\mathcal{F}_\Gamma}_{G\times L}$$

of ∞-categories.

Definition 2.5.7. We denote a classifying space for the graph family $\mathcal{F}_\Gamma$ by $\mathrm{E}_G L \in \mathcal{S}_{G\times L}$. We call $\mathrm{B}_G(L) := (\mathrm{E}_G L)/L \in \mathcal{S}_G$ the *classifying space* for G-equivariant L-principal bundles.

Construction 2.5.8. By [Lur09, 5.2.5.1.], the composite functor

$$(-)/L : \ \mathcal{S}^{\mathcal{F}_\Gamma}_{G\times L} \simeq (\mathcal{S}_{G\times L})_{/\mathrm{E}_G L} \xrightarrow{(-)/L} (\mathcal{S}_G)_{/\,\mathrm{B}_G(L)}$$

is canonically left adjoint to the composite

$$(\mathcal{S}_G)_{/\,\mathrm{B}_G(L)} \xrightarrow{\mathrm{Infl}_G^{G\times L}} (\mathcal{S}_{G\times L})_{/\,\mathrm{Infl}_G^{G\times L}(\mathrm{B}_G(L))} \longrightarrow (\mathcal{S}_{G\times L})_{/\mathrm{E}_G L} \simeq \mathcal{S}^{\mathcal{F}_\Gamma}_{G\times L},$$

where we pull back along the adjunction unit $\mathrm{E}_G L \to \mathrm{Infl}_G^{G\times L}(\mathrm{B}_G(L))$.

Proposition 2.5.9. *The adjunction from Construction 2.5.8 defines an adjoint equivalence*

$$\mathcal{S}^{\mathcal{F}_\Gamma}_{G\times L} \simeq (\mathcal{S}_G)_{/\,\mathrm{B}_G(L)} \qquad \textit{over} \qquad \mathcal{S}_G.$$

Proof. The unit evaluates at $E \in \mathcal{S}^{\mathcal{F}_\Gamma}_{G\times L}$ to the canonical map $E \to (E/L) \times_{\mathrm{B}_G(L)} \mathrm{E}_G L$. By Corollary 2.5.6, this identifies with a morphism of principle bundles over the same base, so it is a homeomorphism. Conversely, if $X \in (\mathrm{Top}_G)^{\mathrm{Lie}(G),\circ}$ and $f : X \to \mathrm{B}_G(L)$ is continuous G-map, we obtain a G-equivariant L-principle bundle $f^*(\mathrm{E}_G L) \to X$. The counit evaluates at f to the induced homeomorphism $(f^*(\mathrm{E}_G L))/L \to X$. □

Construction 2.5.10. The composite equivalence

$$\mathrm{Bun}_{(G,L)} \simeq \mathcal{S}^{\mathcal{F}_\Gamma}_{G\times L} \simeq (\mathcal{S}_G)_{/\,\mathrm{B}_G(L)}$$

of categories over $\mathcal{S}_G$ implies that we can straighten the functor

$$\mathrm{Bun}_{(G,L)} \to \mathcal{S}_G, \qquad (E \to X) \mapsto X$$

to a functor

$$(\mathcal{S}_G)^{\mathrm{op}} \to \mathcal{S}, \qquad X \mapsto \mathrm{Bun}_{(G,L)}(X),$$

which assigns to X, the space

$$\mathrm{Bun}_{(G,L)}(X) := \mathrm{Bun}_{(G,L)} \times_{\mathcal{S}_G} \{X\}$$

of G-equivariant L-principle bundles over X. Moreover, the following holds:

Corollary 2.5.11. *The functor* $\mathrm{Bun}_{(G,L)}(-) : (\mathcal{S}_G)^{\mathrm{op}} \to \mathcal{S}$ *is represented by the classifying space* $\mathrm{B}_G(L) \in \mathcal{S}_G$ *together with the universal* G*-equivariant* L*-principal bundle* $E_G L \to \mathrm{B}_G(L)$*.*

Note that for a $\mathrm{Lie}(G)$-cofibrant space X, the set

$$\mathrm{Bun}_{(G,L)}(X)/_{\mathrm{iso}} := \pi_0 \mathrm{Bun}_{(G,L)}(X)$$

consists of isomorphism classes of G-equivariant L-principal bundles. The π_0-version of Corollary 2.5.11 is the following

Proposition 2.5.12. *Let* $X \in \mathrm{Top}_G$ *be* $\mathrm{Lie}(G)$*-cofibrant. Then,*

$$[X, \mathrm{B}_G(L)]^G \to \mathrm{Bun}_{(G,L)}(X)_{/iso}, \qquad f \mapsto f^*(E_G L)$$

defines a bijection from the set of G*-equivariant homotopy classes of maps to the set of isomorphism classes of* G*-equivariant* L*-principle bundles on* X*.*

2.5.2. *Stages of the Family of Graph Subgroups for Pro Compact Lie Groups.* Let L be a compact Lie group and let morphisms $\varphi_n : G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups. Then, the morphisms $\varphi_n \times \mathrm{id} : G \times L \to G_n \times L$ exhibit the product $G \times L$ as the inverse limit

$$G \times L \cong \varprojlim_n (G_n \times L)$$

of topological groups, with each $G_n \times L$ a compact Lie group. Recall that for any family of subgroups $\mathcal{F} \subseteq \mathrm{Lie}(G \times L)$ we defined the n-th stage of $\mathcal{F}$ in Definition 2.4.16. We are interested in the case that $\mathcal{F} = \mathcal{F}_\Gamma \subseteq \mathrm{Lie}(G \times L)$ is the family of graph subgroups of $G \times L$, in the sense of Definition 2.5.2.

Lemma 2.5.13. The n-th stage of the family of graph subgroups $\mathcal{F}_\Gamma$ of $G \times L$ is the family of graph subgroups $\mathcal{F}_{\Gamma_n}$ of $G_n \times L$.

Proof. Note that $U \leq \mathrm{Lie}(G \times L)$ is a graph subgroup if and only if for all $l \in L$ we have $(1, l) \in U$ implies $l = 1$. Applying the same characterization for graph subgroups of $G_n \times L$, the statement readily follows. □

We conclude from Equation (22), that the unique map

$$\mathrm{colim}_n \mathrm{Infl}_{G_n \times L}^{G \times L} (\mathrm{E}_{G_n} L) \xrightarrow{\simeq} \mathrm{E}_G L \tag{26}$$

in $\mathcal{S}_{G \times L}$ is an equivalence.

Lemma 2.5.14. The Beck-Chevalley transformation of the equivalence

$$\mathrm{Infl}_{G_n \times L}^{G \times L} \circ \mathrm{Infl}_{G_n}^{G_n \times L} \Longrightarrow \mathrm{Infl}_G^{G \times L} \circ \mathrm{Infl}_{G_n}^{G}$$

is an equivalence

$$(-)/L \circ \mathrm{Infl}_{G_n \times L}^{G \times L} \overset{\simeq}{\Longrightarrow} \mathrm{Infl}_{G_n}^{G} \circ (-)/L$$

of functors $\mathcal{S}_{G_n \times L} \to \mathcal{S}_G$.

Proof. Unraveling Construction 2.3.5, we see that the Beck-Chevalley transformation evaluates on the orbit of $K \leq G_n \times L$ to the canonical map

$$G/\left(\mathrm{pr}_G\left((\varphi_n \times \mathrm{id})^{-1}(K)\right)\right) \to G/\varphi_n^{-1}\left(\mathrm{pr}_{G_n}(K)\right)$$

in Orb_G. The inclusion of subgroups inducing that map is an equality. Because all of the above functor preserves colimits, the result follows. □

Applying $(-)/L : \mathcal{S}_{G\times L} \to \mathcal{S}_G$ to the equivalence in Equation (26) yields an equivalence

$$\operatorname{colim}_n \operatorname{Infl}_{G_n}^G(\mathrm{B}_{G_n}(L)) \xrightarrow{\simeq} \mathrm{B}_G(L) \tag{27}$$

of G-spaces, where we employed Lemma 2.5.14.

By Equation (23), the top horizontal arrow in the following diagram

$$\begin{array}{ccc}
\operatorname{colim}^L (\mathcal{S}_{G_n\times L})_{/\mathrm{E}_{G_n}L} & \longrightarrow & (\mathcal{S}_{G\times L})_{/\mathrm{E}_G L} \\
\big\downarrow & & \big\downarrow{\scriptstyle (-)/L} \\
\operatorname{colim}^L (\mathcal{S}_{G_n})_{/\,\mathrm{B}_{G_n}(L)} & \longrightarrow & (\mathcal{S}_G)_{/\,\mathrm{B}_G(L)}
\end{array}$$

is an equivalence in Pr^L. By Proposition 2.5.9, the vertical arrows in the previous diagram, induced by taking quotients by the L-action, are equivalences. The diagram commutes by Lemma 2.5.14, and, therefore, the lower horizontal functor is an equivalence, as well.

2.5.3. *G-universes.* Let G be a compactly metrizable topological group.

Definition 2.5.15. A *complete G-universe $\mathcal{U}_G$* is an orthogonal G-representation of countable dimension, such that, for any finite-dimensional orthogonal G-representation V, any countable infinite direct sum of copies of V isometrically G-linearly embeds into $\mathcal{U}_G$.

Construction 2.5.16. If the topological group G is an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups, then we may construct the countably-dimensional orthogonal G-representation

$$\mathcal{U} := \operatorname{colim}_n \left(\mathcal{U}_{G_0} \to \mathcal{U}_{G_1} \to \mathcal{U}_{G_1} \to \cdots\right), \tag{28}$$

where the morphism $\mathcal{U}_{G_i} \to \mathcal{U}_{G_{i+1}}$ is an isometric and G_{i+1}-linear embedding of an inflated complete G_i-universe $\mathcal{U}_{G_i}$ into a complete G_{i+1}-universe $\mathcal{U}_{G_{i+1}}$. By Theorem 2.1.4, any finite-dimensional orthogonal G-representation is inflated from a G_n-representation for some $n \geq 0$. We conclude that $\mathcal{U}$ is a complete G-universe.

Corollary 2.5.17. *Let $H \leq G$ be a closed subgroup of a compactly metrizable group G. Then any restricted G-universe $\operatorname{Res}_H^G(\mathcal{U}_G)$ is a complete H-universe.*

Proof. If morphisms $\varphi_n : G \to G_n$ exhibit the topological group G as $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups, then the morphism $H \to \varprojlim_n H_n$ is an isomorphism, as well, for $H_n := \varphi_n(H) \leq G_n$, by [RZ00, Corollary 1.1.8.]. Consequently, we may employ Construction 2.5.16 to reduce to the well-known compact Lie group case, see [Sch18, Remark 1.1.13.]. □

Convention 2.5.18. We topologize an infinite-dimensional orthogonal representation as colimit of its finite-dimensional subspaces.

Definition 2.5.19. Let G and L be compactly metrizable topological groups. For an orthogonal L-representation V and an orthogonal G-representation U, we write $\mathbb{L}(V, U)$ for the $(G\times L)$-space of linear isometric embeddings of V into U.

The proof of [GJMS86, Lemma 1.5.] carries over to our situation:

Lemma 2.5.20. Suppose V and U are orthogonal G-representations such that a countable infinite direct sum of copies of V isometrically G-linearly embeds into U. Then, the G-space $\mathbb{L}(V, U)$ is contractible via a G-equivariant homotopy. In particular, a complete G-universe is unique up to contractible choice.

2.5.4. *A Convenient Model for Equivariant Classifying Spaces.*

Lemma 2.5.21 ([Sch18, Proposition 1.1.26.])**.** Let G be a compactly metrizable group. Let V be a finite-dimensional faithful orthogonal representation of a compact Lie group L. Then $\mathbb{L}(V, \mathcal{U}_G)$ is a $\mathcal{F}_\Gamma$-cofibrant replacement of the point $* \in \mathrm{Top}_{G\times L}$.

Proof. We assume that the topological group G is an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups. By [Sch18, Proposition 1.1.26.], the $(L \times G_n)$-space $\mathbb{L}(V, \mathcal{U}_{G_n})$ has the required property. It follows from [Sch18, Proposition 1.1.19.], that $\mathbb{L}(V, \mathcal{U}_{G_n}) \to \mathbb{L}(V, \mathcal{U}_{G_{n+1}})$ is a $\mathrm{Lie}(G \times L)$-cofibration. We conclude that $\mathrm{colim}_n \, \mathbb{L}(V, \mathcal{U}_{G_n}) \cong \mathbb{L}(V, \mathcal{U}_G)$ is $\mathrm{Lie}(G \times L)$-cofibrant. To conclude that the $G \times L$-space $\mathbb{L}(V, \mathcal{U}_G)$ is a classifying space for the family of graph subgroups, we check on fixed point functors, which preserve filtered colimits, to reduce to Schwede's result [Sch18, Proposition 1.1.26.] in the compact Lie group case. □

We conclude that the quotient map $\mathbb{L}(V, \mathcal{U}_G) \to \mathbb{L}(V, \mathcal{U}_G)/L$ models the universal G-equivariant L-principal bundle $\mathrm{E}_G L \to \mathrm{B}_G(L)$.

Example 2.5.22. For $L = O(n)$ an orthogonal group, we deduce that

$$\gamma_n : \mathbb{L}(\mathbb{R}^n, \mathcal{U}_G) \times_{O(n)} \mathbb{R}^n \to \mathrm{Gr}_n(\mathcal{U}_G)$$

models the universal G-equivariant n-dimensional real vector bundle.

Observation 2.5.23. The previous arguments carry over to a unitary complete G-universe $\mathcal{U}_G^{\mathbb{C}}$ and a faithful complex L-representation V.

Example 2.5.24. For $L = U(n)$ an unitary group, we deduce that

$$\gamma_n^{\mathbb{C}} : \mathbb{L}(\mathbb{C}^n, \mathcal{U}_G) \times_{U(n)} \mathbb{C}^n \to \mathrm{Gr}_n(\mathcal{U}_G^{\mathbb{C}})$$

models the universal G-equivariant complex n-dimensional vector bundle.

2.6. **Genuine G-Spectra.** Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim G_n$ with each G_n a compact Lie group. Unraveling Definition 2.2.4, the inflations $\mathrm{Infl}_{G_n}^G : \mathrm{Sp}_{G_n} \to \mathrm{Sp}_G$ induce an equivalence

$$\mathrm{colim}^L \left(\mathrm{Sp}_{G_0} \xrightarrow{\mathrm{Infl}_{G_0}^{G_1}} \mathrm{Sp}_{G_1} \xrightarrow{\mathrm{Infl}_{G_1}^{G_2}} \mathrm{Sp}_{G_2} \xrightarrow{\mathrm{Infl}_{G_2}^{G_3}} \mathrm{Sp}_{G_3} \xrightarrow{\mathrm{Infl}_{G_3}^{G_4}} \mathrm{Sp}_{G_4} \to \cdots \right) \xrightarrow{\simeq} \mathrm{Sp}_G \tag{29}$$

of ∞-categories. Here the colimit is taken in the category Pr^L of presentable ∞-categories and left adjoint functors. The diagram in Equation (29) factors through the subcategory $\mathrm{Pr}^{L,\omega} \subseteq \mathrm{Pr}^L$ spanned by presentable compactly generated categories and left adjoint compact object preserving functors. It follows from [Lur09, Proposition 5.5.7.6.] and the remark in [Lur09, Notation 5.5.7.7.] that $\mathrm{Pr}^{L,\omega} \to \mathrm{Pr}^L$ preserves filtered colimits. By [Lur09, Proposition 5.5.7.8.] and [Lur17, Lemma 7.3.5.10.], taking compact object defines a filtered colimit preserving functor $(-)^\omega : \mathrm{Pr}^{L,\omega} \to \mathrm{Cat}_\infty$ to the category of small categories. We conclude the following:

Lemma 2.6.1. Suppose surjective group homomorphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim G_n$ with each G_n a compact Lie group. Then, the inflations $\mathrm{Infl}_{G_n}^G : \mathrm{Sp}_{G_n}^\omega \to \mathrm{Sp}_G^\omega$ exhibit the category Sp_G^ω of compact G-spectra as the filtered colimit

$$\mathrm{colim} \left(\mathrm{Sp}_{G_0}^\omega \xrightarrow{\mathrm{Infl}_{G_0}^{G_1}} \mathrm{Sp}_{G_1}^\omega \xrightarrow{\mathrm{Infl}_{G_1}^{G_2}} \mathrm{Sp}_{G_2}^\omega \xrightarrow{\mathrm{Infl}_{G_2}^{G_3}} \mathrm{Sp}_{G_3}^\omega \xrightarrow{\mathrm{Infl}_{G_3}^{G_4}} \mathrm{Sp}_{G_4}^\omega \to \cdots \right) \xrightarrow{\simeq} \mathrm{Sp}_G^\omega \tag{30}$$

where the colimit can by computed in the category Cat_∞ of small ∞-categories. Moreover, the inclusion $\mathrm{Sp}_G^\omega \to \mathrm{Sp}_G$ extends to a unique filtered colimit preserving functor $\mathrm{Ind}(\mathrm{Sp}_G^\omega) \to \mathrm{Sp}_G$, which is an equivalence $\mathrm{Ind}(\mathrm{Sp}_G^\omega) \xrightarrow{\simeq} \mathrm{Sp}_G$ of ∞-categories.

Proof. The last statement follows from [Lur09, Proposition 5.5.7.10.]. □

Lemma 2.6.2. Suppose surjective group homomorphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim G_n$ with each G_n a compact Lie group. Let $X \in \mathrm{Sp}_{G_n}^\omega$ be a compact G_n-spectrum. Then, for any G_n-spectrum $Y \in \mathrm{Sp}_{G_n}$, the inflations induce an equivalence

$$\operatorname*{colim}_{m \geq n} \mathrm{map}_{\mathrm{Sp}_{G_m}}(\mathrm{Infl}_{G_n}^{G_m}(X), \mathrm{Infl}_{G_n}^{G_m}(Y)) \to \mathrm{map}_{\mathrm{Sp}_G}(\mathrm{Infl}_{G_n}^{G}(X), \mathrm{Infl}_{G_n}^{G}(Y))$$

of mapping spectra.

Proof. By passing to filtered colimits in Y, we may assume that Y is compact, as well. Moreover, it suffices to prove the result on mapping spaces, because we may suspend Y to get to the negative homotopy groups of the mapping spectrum. The result follows by applying the formula for mapping spaces in filtered colimits of ∞-categories, see [BBB24, Lemma 2.3.], to the description of compact G-spectra in Equation (30). □

Construction 2.6.3. Recall from [LNP25] that the functor $\mathrm{Sp}_\bullet : \mathrm{CptLie}^{\mathrm{op}} \to \mathrm{Pr}^L$ lifts to a functor $\mathrm{Sp}_\bullet^\otimes : \mathrm{CptLie}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, encoding the smash product of equivariant spectra. The forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}) \to \mathrm{Pr}^L$ preserves filtered colimits. Thus, the unique functor

$$\mathrm{Sp}_\bullet^\otimes : \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

extending the functor $\mathrm{Sp}_\bullet^\otimes : \mathrm{CptLie}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ from [LNP25] in a filtered colimit preserving way lifts the functor $\mathrm{Sp}_\bullet : \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \to \mathrm{Pr}^L$. In particular, we may view Equation (29) as a colimit diagram in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$.

Notation 2.6.4. For a compactly metrizable topological group G we fix the following notation:

- The smash product of G-spectra, constructed in Construction 2.6.3, is denoted by

  $$(-) \otimes (-) : \ \mathrm{Sp}_G \times \mathrm{Sp}_G \to \mathrm{Sp}_G\,.$$

  The tensor unit of G-spectra is denoted by

  $$\mathbb{S} := \mathrm{Infl}_1^G(\mathbb{S}) \in \mathrm{Sp}_G\,.$$

- Following Convention 2.2.6, we use the same notation for inflation and restriction, as well as their right adjoints, as we did in the unstable setting.
- For $H \in \mathrm{Sub}(G)$, we call the composite

  $$(-)^H : \mathrm{Sp}_G \xrightarrow{\mathrm{Res}_H^G} \mathrm{Sp}_H \xrightarrow{\mathrm{map}_{\mathrm{Sp}_H}(\mathbb{S},-)} \mathrm{Sp}$$

  categorical H-fixed points.
- Naturally in $G \in \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}}$, we obtain a natural symmetric monoidal functor

  $$\Sigma^\infty : \mathcal{S}_{G,*} \to \mathrm{Sp}_G$$

  by extending the natural transformation $\Sigma^\infty : \mathcal{S}_{\bullet,*} \to \mathrm{Sp}_{\bullet,*}$ of functors $\mathrm{CptLie}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ from $\mathrm{CptLie}^{\mathrm{op}}$ to its Ind-category $\mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}}$, see [LNP25, Remark 10.7.]. By this construction, restriction-inflation commutes with suspension Σ^∞. Moreover, the suspension functor Σ^∞ preserves compact objects, as the whole construction factors through $\mathrm{CAlg}(\mathrm{Pr}^{L,\omega})$.

For a compact Lie group G, the smallest subcategory of Sp_G containing $\{\Sigma^\infty_+ G/H\}_{H\in \mathrm{Lie}(G)}$, which is closed under shifts and small colimits, is Sp_G, itself. From Lemma 2.6.1, we deduce that the same holds for all $G \in \mathrm{Grp}(\mathrm{CptMet})$. Because Sp_G is presentable, this property is equivalent to the following:

Proposition 2.6.5 (Generation By Orbits)**.** *For any compactly metrizable group G, the set $\{\Sigma^\infty_+ G/H\}_{H\in \mathrm{Lie}(G)}$ of compact objects generates Sp_G, in the sense that for all $X \in \mathrm{Sp}_G$*

$$\mathrm{map}_{\mathrm{Sp}_G}(\Sigma^\infty_+ G/H, X) \simeq 0 \in \mathrm{Sp} \text{ for all } H \in \mathrm{Lie}(G) \qquad \textit{implies} \qquad X \simeq 0 \in \mathrm{Sp}_G\,. \tag{31}$$

Definition 2.6.6. Recall that a category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is called *rigidly compactly generated*, if $\mathcal{C}$ is compactly generated and an object $X \in \mathcal{C}$ is dualizable if and only if $X \in \mathcal{C}^\omega$ is compact. We write Rig^L for the full subcategory of $\mathrm{CAlg}(\mathrm{Pr}^L)$ consisting of rigidly compactly generated categories.

Note that a morphism in Rig^L preserves dualizable and thus compact objects.

Lemma 2.6.7. The functor $\mathrm{Sp}^\otimes_\bullet : \mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ factors through Rig^L.

Proof. Let G be a compactly metrizable group. Because the unit $\mathbb{S} \simeq \Sigma^\infty_+(*) \in \mathrm{Sp}_G$ is compact, all dualizable G-spectra are compact. In light of Proposition 2.6.5, it suffices to see that all suspensions of orbits $\Sigma^\infty_+ G/H$ of closed subgroups $H \leq G$ with $H \in \mathrm{Lie}(G)$ are dualizable. The G-spectrum $\Sigma^\infty_+ G/H$ is inflated from the suspension of an orbit $\Sigma^\infty_+ G_n/H_n$ of some $H_n \in \mathrm{Sub}(G_n)$ along a surjective morphism $G \to G_n$ to a compact Lie group G_n. As the inflation functor $\mathrm{Infl}^G_{G_n}(-)$ is symmetric monoidal, the result follows from the compact Lie group case, see [GJMS86, Chapter III.]. □

2.6.1. *Inverting Representation Spheres.*

Remark 2.6.8. Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $\varprojlim_n G_n$ of compact Lie groups G_n. For a d-dimensional real G-representation V, we may choose a G-invariant inner product on V, by compactness of G. By Theorem 2.1.4, the group homomorphism $V : G \to O(d)$ encoding matrices of the G-action on an orthogonal basis of V factors as

$$G \to G_n \xrightarrow{W} O(d)$$

for some $n \in \mathbb{N}$ and some G_n-representation $W : G_n \to O(d)$. Hence, the topological G-spaces S^V and $\mathrm{Infl}^G_{G_n}(S^W)$, obtained by one point compactification of V and W, respectively, agree.

Definition 2.6.9 (Representation Sphere)**.** For a compactly metrizable group G and a finite dimensional G-representation V, we define the *representation sphere* $S^V \in \mathcal{S}_{G,*}$ as the genuine G-equivariant homotopy type of the one point compactification $S^V \in \left(\mathrm{Top}_{G,*}\right)^{\mathrm{Lie}(G),\circ}$ of V.

Proposition 2.6.10 (Universal Property of Sp_G)**.** *Let G be a compactly metrizable group and $(\mathcal{C}, \otimes) \in \mathrm{CAlg}(\mathrm{Pr}^L)$. Any symmetric monoidal left adjoint functor $\mathcal{S}_{G,*} \to \mathcal{C}$, which sends the representation sphere of any finite dimensional G-representation to an $\otimes$-invertible object, factors uniquely through the suspension functor $\Sigma^\infty : \mathcal{S}_{G,*} \to \mathrm{Sp}_G$.*

The compact Lie group version of this Proposition is [GM23, Corollary C.7.].

Proof. We may assume that the topological group G is an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $G = \varprojlim G_n$ along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups G_n. For any compactly metrizable group K, we denote by $\mathrm{Map}^{\mathcal{U}_K^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{K,*}, \mathcal{C})$ the subspace of the mapping space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{K,*}, \mathcal{C})$ containing those functors sending all representation spheres to $\otimes$-invertible objects. We denote by $\mathrm{pic}(\mathcal{C}) \subseteq \pi_0(\mathcal{C}^\simeq)$ the subset of the set of equivalence classes of objects of $\mathcal{C}$ consisting of $\otimes$-invertible objects. Here $\mathcal{C}^{\simeq} := \mathrm{Map}_{\mathrm{Cat}_\infty}(*, \mathcal{C})$ is the core of $\mathcal{C}$. When $s(\mathcal{U}_K)$ denotes the poset of finite dimensional subrepresentations of a complete K-universe $\mathcal{U}_K$, then the diagram

$$\begin{array}{ccc}
\mathrm{Map}^{\mathcal{U}_K^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{K,*}, \mathcal{C}) & \xrightarrow{\mathrm{ev}(S^V)} & \prod_{V \in s(\mathcal{U}_K)} \mathrm{pic}(\mathcal{C}) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{K,*}, \mathcal{C}) & \xrightarrow{\mathrm{ev}(S^V)} & \prod_{V \in s(\mathcal{U}_K)} \pi_0\, \mathcal{C}^{\simeq}
\end{array}$$

is a pullback square. Following Construction 2.5.16, we may write the poset $s(\mathcal{U}_G)$ as nested union of the posets $s(\mathcal{U}_{G_n})$. We conclude that the two right diagonal arrows in the following commutative diagram

$$\begin{array}{ccccc}
\varprojlim_n \mathrm{Map}^{\mathcal{U}_{G_n}^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G_n,*}, \mathcal{C}) & & \longrightarrow & & \varprojlim_n \prod_{V \in s(\mathcal{U}_{G_n})} \mathrm{pic}(\mathcal{C}) \\
 & \nwarrow & & \nearrow \simeq & \\
\big\downarrow & \mathrm{Map}^{\mathcal{U}_G^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G,*}, \mathcal{C}) & \to & \prod_{V \in s(\mathcal{U}_G)} \mathrm{pic}(\mathcal{C}) & \big\downarrow \\
 & \big\downarrow & & \big\downarrow & \\
 & \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G,*}, \mathcal{C}) & \to & \prod_{V \in s(\mathcal{U}_G)} \pi_0(\mathcal{C}^{\simeq}) & \\
 & \swarrow \simeq & & \searrow \simeq & \\
\varprojlim_n \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G_n,*}, \mathcal{C}) & & \longrightarrow & & \varprojlim \prod_{V \in s(\mathcal{U}_{G_n})} \pi_0\, \mathcal{C}^{\simeq}
\end{array}$$

are equivalences. Since the inflations exhibit $\mathcal{S}_{G,*}$ as the colimit $\mathrm{colim}^L_n\, \mathcal{S}_{G_n,*}$ in $\mathrm{CAlg}(\mathrm{Pr}^L)$, the lower left diagonal arrow in the above diagram is an equivalence, as well. Because pullbacks are closed under limits, the large square in the above diagram is a pullback square. We conclude that the lower horizontal morphism in the following commutative square

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathrm{Sp}_G, \mathcal{C}) & \longrightarrow & \varprojlim_n \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathrm{Sp}_{G_n,*}, \mathcal{C}) \\
\big\downarrow (\Sigma^\infty)^* & & (\Sigma^\infty)^* \big\downarrow \\
\mathrm{Map}^{\mathcal{U}_G^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G,*}, \mathcal{C}) & \xrightarrow{\simeq} & \varprojlim_n \mathrm{Map}^{\mathcal{U}_{G_n}^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G_n,*}, \mathcal{C})
\end{array}$$

is an equivalence. The right vertical map in this square is an equivalence by the compact Lie group version of this Proposition 2.6.10, see [GM23, Corollary C.7.]. The top horizontal map is an equivalence because the inflations exhibit Sp_G as the colimit $\mathrm{colim}^L_n\, \mathrm{Sp}_{G_n}$ in $\mathrm{CAlg}(\mathrm{Pr}^L)$. We conclude that the left vertical map, i.e. precomposition by the suspension functor $\Sigma^\infty : \mathcal{S}_{G,*} \to \mathrm{Sp}_G$, defines an equivalence

$$(\Sigma^\infty)^* : \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_G, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Map}^{\mathcal{U}_G^{-1}}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{S}_{G,*}, \mathcal{C}),$$

as well. □

Notation 2.6.11. For a compactly metrizable group G, we denote by $s(\mathcal{U}_G)$ the poset of finite dimensional subrepresentations of a complete G-universe as in Definition 2.5.15. For $V \in s(\mathcal{U}_G)$ we write $\mathbb{S}^{-V} \in \mathrm{Sp}_G$ for the dual ($\otimes$-inverse) of the representation sphere $\mathbb{S}^V := \Sigma^\infty S^V$ of V. We abbreviate

$$\Sigma^{\infty \pm V} := \mathbb{S}^{\pm V} \otimes \Sigma^\infty(-) : \mathcal{S}_{G,*} \to \mathrm{Sp}_G\,.$$

Proposition 2.6.12. *The functors $\Sigma^{\infty-V} : \mathcal{S}_{G,*} \to \mathrm{Sp}_G$ exhibit Sp_G as the colimit of a functor*

$$F : s(\mathcal{U}_G) \to \mathrm{Pr}^L, \qquad V \mapsto \mathcal{S}_{G,*}$$

with transition maps $S^U \wedge (-) : \mathcal{S}_{G,*} \to \mathcal{S}_{G,*}$ for $W \subseteq V \subseteq \mathcal{U}_G$ and $V = U \oplus W$.

Proof. Analogously to [GM23, Appendix C.1] we define the required functor

$$F : s(\mathcal{U}_G) \to \mathrm{Pr}^L,$$

working with the relative category description for $\mathcal{S}_{G,*}$ from Proposition 2.4.19. Then, the same arguments as in [GM23, Lemma C.5., Proposition C.6, Corollary C.7] imply the existence of a presentably symmetric monoidal structure of the component inclusion

$$\mathcal{S}_{G,*} \to \operatorname{colim} F =: \ \mathcal{S}_{G,*}[(S^V)^{-1} : V \in s(\mathcal{U}_G)]$$

at the zero representation $0 \in s(\mathcal{U}_G)$, which satisfies the same universal property as the suspension functor

$$\Sigma^\infty : \mathcal{S}_{G,*} \to \mathrm{Sp}_G$$

satisfies by Proposition 2.6.10. □

For a compactly metrizable group G, the diagram F in Proposition 2.6.12 factors through the subcategory $\mathrm{Pr}^{L,\omega} \subseteq \mathrm{Pr}^L$ spanned by presentable compactly generated categories and left adjoint compact object preserving functors. It follows from [Lur09, Proposition 5.5.7.6.] and the remark in [Lur09, Notation 5.5.7.7.] that $\mathrm{Pr}^{L,\omega} \to \mathrm{Pr}^L$ preserves filtered colimits. By [Lur09, Proposition 5.5.7.8.] and [Lur17, Lemma 7.3.5.10.], taking compact object defines a filtered colimit preserving functor $(-)^\omega : \mathrm{Pr}^{L,\omega} \to \mathrm{Cat}_\infty$ to the category of small categories. We conclude the following:

Corollary 2.6.13. *The functors $\Sigma^{\infty-V} : \mathcal{S}^\omega_{G,*} \to \mathrm{Sp}^\omega_G$ exhibit the category of compact G-spectra Sp^ω_G as the colimit of the following functor*

$$F^\omega : s(\mathcal{U}_G) \to \mathrm{Cat}_\infty, \qquad V \mapsto \mathcal{S}^\omega_{G,*}$$

to the category of small ∞-categories. For an inclusion $W \subseteq V \subseteq \mathcal{U}_G$ the diagram F^ω has the transition map $S^U \wedge (-) : \mathcal{S}^\omega_{G,*} \to \mathcal{S}^\omega_{G,*}$ for $V = U \oplus W$.

Corollary 2.6.14. *Let G be a compactly-metrizable group. For any two compact pointed G-spaces $X, Y \in \mathcal{S}^\omega_{G,*}$ the functors $\Sigma^{\infty-V}$ induce an equivalence*

$$\operatorname*{colim}_{V \in s(\mathcal{U}_G)} \mathrm{Map}_{\mathcal{S}_{G,*}}(S^V \wedge X, S^V \wedge Y) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Sp}_G}(\Sigma^\infty X, \Sigma^\infty Y) \tag{32}$$

on mapping spaces. The same holds for not necessarily compact $Y \in \mathcal{S}_{G,*}$.

Proof. The statement follows from Corollary 2.6.13 by the mapping space formula for filtered colimits of small ∞-categories, see [BBB24, Lemma 2.3.]. The second statement follows because both sides of the equation are stable under filtered colimits in Y. □

Corollary 2.6.15. *Let G be a compactly-metrizable group. Let $\mathrm{Sp}^{\mathrm{orth}}_G$ denote the underlying ∞-category of Fausk's model [Fau08] for G-spectra with respect to the family $\mathrm{Lie}(G)$. Let*

$$\Sigma^{\infty,\mathrm{orth}} : \ \left(\mathrm{Top}_{G,*}\right)^{\mathrm{Lie}(G),\circ} \to \mathrm{Sp}^{\mathrm{orth}}_G$$

denote the functor of ∞-categories obtained from the left Quillen suspension ∞-functor of [Fau08]. There exists a unique symmetric monoidal left adjoint functor $\mathrm{Sp}_G \to \mathrm{Sp}^{\mathrm{orth}}_G$ making the diagram

$$\begin{array}{ccc}
\mathrm{Sp}_G & \longrightarrow & \mathrm{Sp}^{\mathrm{orth}}_G \\
{\scriptstyle \Sigma^\infty}\uparrow & & \uparrow{\scriptstyle \Sigma^{\infty,\mathrm{orth}}} \\
\mathcal{S}_{G,*} & \xrightarrow[(2.4.19)]{\simeq} & \left(\mathrm{Top}_{G,*}\right)^{\mathrm{Lie}(G),\circ}
\end{array}$$

commute. This functor $\mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{orth}}$ *is an equivalence of* ∞*-categories.*

In the case that G is a profinite group, this Corollary 2.6.15 is due to [BBB24].

Proof. Existence and uniqueness of the functor $\mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{orth}}$ follow from Proposition 2.6.10. By our Corollary 2.6.14 and Fausk's [Fau08, Corollary 7.2.], the functor $\mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{orth}}$ is fully-faithful on suspension spectra of compact pointed G-spaces. Because Sp_G is stable and $\mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{orth}}$ preserves finite limits and colimits, the functor $\mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{orth}}$ is consequently fully-faithful, on all finite colimits of suspension spectra. By Fausk's [Fau08, Theorem 4.4.], the category $\mathrm{Sp}_G^{\mathrm{orth}}$ is generated, in the sense of stable categories, by the suspension spectra of the orbits $\{G/U_+\}_{U \in \mathrm{Lie}(G)}$. Moreover, these are compact objects in $\mathrm{Sp}_G^{\mathrm{orth}}$. The analog statement also holds for the category Sp_G of G-spectra by Proposition 2.6.5. We conclude, that $\mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{orth}}$ induces an equivalence on compact objects, and the result follows by taking Ind-categories. □

Proposition 2.6.16. *Let G be a compactly metrizable group and $H \le G$ a subgroup with $H \in \mathrm{Lie}(G)$. Then, the functor $\mathrm{Res}_H^G : \mathrm{Sp}_G \to \mathrm{Sp}_H$ admits a left adjoint $\mathrm{Ind}_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$. The Beck-Chevalley transformation of the natural equivalence*

$$\Sigma^\infty \circ \mathrm{Res}_H^G \Rightarrow \mathrm{Res}_H^G \circ \Sigma^\infty$$

defines an equivalence

$$\mathrm{Ind}_H^G \circ \Sigma^\infty \overset{\simeq}{\Longrightarrow} \Sigma^\infty \circ \mathrm{Ind}_H^G \tag{33}$$

of functors $\mathcal{S}_{H,*} \to \mathrm{Sp}_G$.

Proof. By Corollary 2.6.13, the functors $\Sigma^{\infty - V} : \mathcal{S}_{G,*}^\omega \to \mathrm{Sp}_G^\omega$ exhibit Sp_G^ω as colimit over the poset $s(\mathcal{U}_G)$ in Cat_∞ . By Corollary 2.5.17, $\mathrm{Res}_H^G(\mathcal{U}_G)$ is a complete H-universe. So, again, by Corollary 2.6.13, the functors

$$\Sigma^{\infty - \mathrm{Res}_H^G(V)} : \mathcal{S}_{H,*}^\omega \to \mathrm{Sp}_H^\omega$$

exhibit Sp_H^ω as the colimit over the poset $s(\mathcal{U}_G)$ in Cat_∞. By the second part of [GM23, Theorem C.6.], the restriction functor $\mathrm{Res}_H^G \colon \mathrm{Sp}_G^\omega \to \mathrm{Sp}_H^\omega$ is the morphism induced on colimits by a natural transformation $\mathrm{Stab}(\mathrm{Res})$ of functors $s(\mathcal{U}_G) \to \mathrm{Cat}_\infty$. This natural transformation $\mathrm{Stab}(\mathrm{Res})$ evaluates at each G-representation $V \subseteq \mathcal{U}_G$ the restriction functor $\mathrm{Res}_H^G \colon \mathcal{S}_{G,*}^\omega \to \mathcal{S}_{H,*}^\omega$ between pointed G- and H-spaces.

We claim that the Beck-Chevalley transformation of the natural equivalence $\mathrm{Res}_H^G(S^V) \wedge (-) \Rightarrow \mathrm{Res}_H^G(S^V \wedge (-))$ defines an equivalence

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(S^V) \wedge (-)) \Longrightarrow S^V \wedge \mathrm{Ind}_H^G(-)$$

for all $V \subseteq \mathcal{U}_G$. By Proposition 2.4.19, we may check the Beck-Chevalley condition on cofibrant objects in the model $(\mathrm{Top}_G)^{\mathrm{Lie}(G),\circ}$. Here, we obtain the homeomorphism, which in the literature is often referred to as shearing isomorphism, see [Sch18, page 262].

Because this Beck-Chevalley condition holds, we may employ [Lur17, Proposition 7.3.2.11], to obtain a natural transformation[6] $\mathrm{Stab}(\mathrm{Ind}_H^G)$ of functors $s(\mathcal{U}_G) \to \mathrm{Cat}_\infty$ together with the following structure: The natural transformations $\mathrm{Stab}(\mathrm{Ind}_H^G)$ and $\mathrm{Stab}(\mathrm{Res}_H^G)$ unstraighten to relative adjoint morphisms of coCartesian fibrations over $s(\mathcal{U}_G)$, in the sense of [Lur17, Proposition 7.3.2.1]. Moreover, the relative adjunction restricts on each $V \in s(\mathcal{U}_G)$ to the induction $\mathrm{Ind}_H^G : \mathcal{S}_{H,*}^\omega \to \mathcal{S}_{G,*}^\omega$ and restriction $\mathrm{Res}_H^G : \mathrm{Sp}_{G,*}^\omega \to \mathrm{Sp}_{H,*}^\omega$ adjunction.

Dwyer-Kan localization is a $(\infty, 2)$-functor by [Lur17, Proposition 4.1.7.2.], so, if we localize the coCartesian unstraightenings at coCartesian edges, to compute the colimits Sp_H^ω and Sp_G^ω in

[6] We additionally use that a relative left adjoint preserve coCartesian edges by the opposite of [Lur17, Proposition 7.3.2.6.], and therefore straightens to the natural transformations $\mathrm{Stab}(\mathrm{Ind}_H^G)$.

Cat_∞, the relative adjunction between the coCartesian unstraigthenings is sent to an adjunction. Therefore, the natural transformation $\mathrm{Stab}(\mathrm{Ind}_H^G)$ induces a morphism on colimits, denoted $\mathrm{Ind}_H^G : \mathrm{Sp}_H^\omega \to \mathrm{Sp}_G^\omega$, which is left adjoint to $\mathrm{Res}_H^G : \mathrm{Sp}_G^\omega \to \mathrm{Sp}_H^\omega$. Passing to $\mathrm{Ind}(-)$-categories, we obtain a functor $\mathrm{Ind}_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$, which is left adjoint to $\mathrm{Res}_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$. Indeed, taking $\mathrm{Ind}(-)$-categories is an $(\infty, 2)$-functor by the proof of [BB24, Proposition 2.4.3.] and, in particular, preserves adjunctions.

To see whether the Beck-Chevalley transformation in Equation (33) is an equivalence, we can check on compact objects. We restrict the triangle identity of the above relative adjunction to the fiber over $0 \in s(\mathcal{U}_G)$ and localize at the coCartesian edges to obtain the result. □

Remark 2.6.17 (Homotopy Groups)**.** It follows that for $H \le G$ a subgroup of a compactly metrizable group with $H \in \mathrm{Lie}(G)$ the categorical H-fixed point functor $(-)^H : \mathrm{Sp}_G \to \mathrm{Sp}$ is corepresented by $\Sigma^\infty_+(G/H) \simeq \mathrm{Ind}_H^G(\mathbb{S})$. The homotopy group functors

$$\{\pi_*^H(-)\}_{H \in \mathrm{Lie}(G)} \qquad \text{for} \qquad \pi_*^H(-) := \pi_*((-)^H) : \mathrm{Sp}_G \to \mathrm{Ab}^{\mathrm{gr}}$$

to graded abelian groups are jointly conservative by Proposition 2.6.5.

Corollary 2.6.18. *Suppose surjective morphisms $\varphi_n : G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim G_n$ with each G_n a compact Lie group. For $n \ge 0$ and $H \le G$ a subgroup with $H \in \mathrm{Lie}(G)$, the inflations $\{\mathrm{Infl}_{G_m}^G(-)\}_{m \ge n}$ induce an equivalence of lax symmetric monoidal functors $\mathrm{Sp}_{G_n} \to \mathrm{Sp}$,*

$$\operatorname*{colim}_{m \ge n} \left(\mathrm{Infl}_{G_n}^{G_m}(-)\right)^{H_m} \to \left(\mathrm{Infl}_{G_n}^{G}(-)\right)^{H},$$

where $H_m := \varphi_m(H) \le G_m$ is a closed subgroup of G_m. By applying the homotopy group functor $\pi_* : \mathrm{Sp} \to \mathrm{Ab}^{\mathrm{gr}}$, we obtain a preferred (graded ring) isomorphism

$$\operatorname*{colim}_{m \ge n} \pi_*^{H_m}\left(\mathrm{Infl}_{G_n}^{G_m}(E)\right) \to \pi_*^H\left(\mathrm{Infl}_{G_n}^{G}(E)\right)$$

naturally in the (homotopy ring) G_n-spectrum $E \in \mathrm{Sp}_{G_n}$.

Proof. The fixed point functor is co-represented by $G/H \in \mathrm{Orb}_G$. We assumed that $H \in \mathrm{Lie}(G)$, so we may choose an $m \ge n$ such that the canonical map $G/H \to \mathrm{Infl}_{G_m}^G(G_m/H_m)$ is an equivalence. The result follows from Lemma 2.6.2. □

Lemma 2.6.19. Let G be a compactly metrizable group and $H \le G$ a subgroup with $H \in \mathrm{Lie}(G)$. Let $\varphi_n : G \to G_n$ be a surjective continuous group homomorphism and set $H_n := \varphi_n(H) \le G_n$. If $\ker(\varphi_n) \le H$, then, the Beck-Chevalley transformation of the natural equivalence

$$\mathrm{Infl}_{H_n}^G \circ \mathrm{Res}_{H_n}^{G_n} \xRightarrow{\simeq} \mathrm{Res}_H^G \circ \mathrm{Infl}_{G_n}^G$$

is an equivalence

$$\mathrm{Ind}_H^G \circ \mathrm{Infl}_{H_n}^H \xRightarrow{\simeq} \mathrm{Infl}_{G_n}^G \circ \mathrm{Ind}_{H_n}^{G_n}$$

of functors $\mathrm{Sp}_{H_n} \to \mathrm{Sp}_G$.

Proof. The above functors preserve colimits and Sp_{H_n} is generated under colimits and shifts by suspension spectra, see Proposition 2.6.5. Thus, it suffices to check that the Beck-Chevalley transformation evaluates to an equivalence on suspension spectra. Via Proposition 2.6.16, it is sufficient to prove the analogue statement for the same-named functors on the space level. This we have already done in Lemma 2.3.9. □

2.6.2. *Geometric Fixed Points.*

Definition 2.6.20. Let G be a compactly metrizable group and $H \leq G$ a closed subgroup. By Proposition 2.6.10 there exists a unique symmetric monoidal left adjoint functor $\Phi^H_G : \mathrm{Sp}_G \to \mathrm{Sp}$ such that $\Phi^H_G \circ \Sigma^\infty \simeq \Sigma^\infty \circ (-)^H$. This *geometric fixed point functor* factors as $\Phi^H_G \simeq \Phi^H \circ \mathrm{Res}^G_H$, where $\Phi^H : \mathrm{Sp}_H \to \mathrm{Sp}$ is the *absolute geometric fixed point functor.*

Observation 2.6.21. Let $\varphi : G \to G_n$ be a surjective continuous homomorphism of compactly metrizable groups. Let $H \leq G$ be a closed subgroup and $H_n := \varphi(H)$ its image. Then we obtain a preferred equivalence

$$\Phi^H_G \circ \mathrm{Infl}^G_{G_n} \simeq \Phi^{H_n}_{G_n} \tag{34}$$

of symmetric monoidal functors $\mathrm{Sp}_{G_n} \to \mathrm{Sp}$.

Indeed, the equivalence $(\mathrm{Infl}^G_{G_n}(-))^H \simeq (-)^{H_n}$ of functors $\mathcal{S}_{G_n,*} \to \mathcal{S}_*$ from (the based version) of Corollary 2.6.18, provides a preferred symmetric monoidal equivalence

$$\Phi^H_G \circ \mathrm{Infl}^G_{G_n} \circ \Sigma^\infty \simeq \Sigma^\infty \circ (-)^H \circ \mathrm{Infl}^G_{G_n} \simeq \Sigma^\infty \circ (-)^{H_n} \simeq \Phi^{H_n}_{G_n} \circ \Sigma^\infty$$

of functors $\mathcal{S}_{G_n,*} \to \mathrm{Sp}$. By the universal property of G_n-spectra, see Proposition 2.6.10, there exists a unique equivalence of symmetric monoidal functors $\Phi^H_G \circ \mathrm{Infl}^G_{G_n} \simeq \Phi^{H_n}_{G_n}$ lifting the previous equivalence along precomposition with $\Sigma^\infty : \mathcal{S}_{G_n,*} \to \mathrm{Sp}_{G_n}$.

Remark 2.6.22. Suppose morphisms $\varphi_n : G \to G_n$ exhibit a topological group G as inverse limit $G \cong \varprojlim G_n$ along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups G_n. Then, the relation in Equation (34) uniquely determines the geometric fixed point functor Φ^H_G for any closed subgroup $H \leq G$. In fact, we obtain an equivalence of symmetric monoidal functors

$$\operatorname*{colim}_{n\in\mathbb{N}}{}^L \, \Phi^{H_n}_{G_n} \simeq \Phi^H_G : \operatorname*{colim}_{n\in\mathbb{N}}{}^L \, \mathrm{Sp}_{G_n} \simeq \mathrm{Sp}_G \longrightarrow \mathrm{Sp},$$

where the colimit along inflations is computed in $\mathrm{CAlg}(\mathrm{Pr}^L)$.

Lemma 2.6.23. Let G be a compactly metrizable group and $X, Y \in \mathcal{S}^\omega_{G,*}$ pointed compact G-spaces. The inclusion $s(\mathcal{U}^G_G) \to s(\mathcal{U}_G)$ of the subposet spanned by G-fixed representations admits a right adjoint $V \mapsto V^G$ and the diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Sp}_G}(\Sigma^\infty Y, \Sigma^\infty X) & \xrightarrow{\Phi^G} & \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty Y^G, \Sigma^\infty X^G) \\
\uparrow {\scriptstyle (\Sigma^{\infty - V})_V} & & \uparrow {\scriptstyle (\Sigma^{\infty - V^G})_V} \\
\operatorname*{colim}_{V \in s(\mathcal{U}_G)} \mathrm{Map}_{\mathcal{S}_{G,*}}(S^V \wedge Y, S^V \wedge X) & \longrightarrow & \operatorname*{colim}_{V \in s(\mathcal{U}^G_G)} \mathrm{Map}_{\mathcal{S}_*}(S^{V^G} \wedge Y^G, S^{V^G} \wedge X^G) \\
& \searrow^{(-)^G} & \downarrow \simeq \\
& & \operatorname*{colim}_{V \in s(\mathcal{U}_G)} \mathrm{Map}_{\mathcal{S}_*}(S^{V^G} \wedge Y^G, S^{V^G} \wedge X^G)
\end{array}$$

commutes, where both upper vertical arrows are equivalences. Moreover, this Lemma also holds for non-compact $X \in \mathcal{S}_{G,*}$.

Proof. It is explained in [GM23, Theorem C.6.] how the universal property of inverting representation spheres applies in practice. This applies in our situation by Proposition 2.6.10. Unraveling our construction of the geometric fixed point functor, the above square commutes by the second part of [GM23, Theorem C.6.], at least for compact $X \in \mathcal{S}^\omega_{G,*}$. Passing to filtered colimits, the general case follows. The vertical arrows in the diagram are equivalences by Corollary 2.6.14. $\square$

2.6.3. *A Convenient Model for Geometric Fixed Points.* For this Section 2.6.3, we fix a compactly metrizable group G. Recall from Example 2.4.2 that $\mathcal{P}_G \subseteq \mathrm{Lie}(G)$ denotes the family of subgroups $H \in \mathrm{Lie}(G)$ with $H \neq G$. We discussed the classifying space $E\mathcal{P}_G \in \mathcal{S}_G$ of this family in Section 2.4. Moreover, we defined the pointed G-space $\widetilde{E\mathcal{P}_G}$ as the cofiber of the G-map $(E\mathcal{P}_G)_+ \to S^0$.

Since $\Phi^G(\Sigma^\infty_+ E\mathcal{P}_G) \simeq *$, the map $\mathbb{S} \to \widetilde{E\mathcal{P}_G} := \Sigma^\infty \widetilde{E\mathcal{P}_G}$ induces an equivalence

$$\Phi^G(X) \xrightarrow{\simeq} \Phi^G(\widetilde{E\mathcal{P}_G} \otimes X)$$

on geometric fixed points for any G-spectrum X.

Proposition 2.6.24. *Let G be a compactly metrizable group. Then, the geometric fixed point functor induces an equivalence*

$$\Phi^G : \mathrm{map}_{\mathrm{Sp}_G}(Y, \widetilde{E\mathcal{P}_G} \otimes X) \to \mathrm{map}_{\mathrm{Sp}}(\Phi^G(Y), \Phi^G(\widetilde{E\mathcal{P}_G} \otimes X)) \simeq \mathrm{map}_{\mathrm{Sp}}(\Phi^G(Y), \Phi^G(X)) \quad (35)$$

on mapping spectra, for any two G-spectra X and Y. The equivalence

$$\Phi^G(\widetilde{E\mathcal{P}_G} \otimes \mathrm{Infl}_1^G(X)) \xRightarrow{\simeq} X$$

serves as a counit for and adjunction $\Phi^G \dashv \widetilde{E\mathcal{P}_G} \otimes \mathrm{Infl}_1(-)$.

Proof. By Example 2.4.15, taking fixed points induces an equivalence of mapping spaces

$$(-)^G : \mathrm{Map}_{\mathcal{S}_{G,*}}(S^V \wedge Y, S^V \wedge (\widetilde{E\mathcal{P}_G} \wedge X)) \to \mathrm{Map}_{\mathcal{S}_*}(S^{V^G} \wedge Y^G, S^{V^G} \wedge (\widetilde{E\mathcal{P}_G} \wedge X)^G)$$

for any pair $X, Y \in \mathcal{S}_{G,*}$ of pointed G-spaces. When Y is compact, we may apply Lemma 2.6.23 to conclude that

$$\Phi^G : \mathrm{Map}_{\mathrm{Sp}_G}(\Sigma^\infty Y, \Sigma^\infty(\widetilde{E\mathcal{P}_G} \wedge X)) \to \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty Y^G, \Sigma^\infty(\widetilde{E\mathcal{P}_G} \wedge X)^G)$$

is an equivalence of spaces. By suspending X, we conclude that the same holds on mapping spectra, i.e. the map

$$\Phi^G : \mathrm{map}_{\mathrm{Sp}_G}(\Sigma^\infty Y, \widetilde{E\mathcal{P}_G} \otimes \Sigma^\infty X) \to \mathrm{map}_{\mathrm{Sp}}(\Phi^G(\Sigma^\infty Y), \Phi^G(\Sigma^\infty(\widetilde{E\mathcal{P}_G} \wedge X))$$

is an equivalence for any compact pointed G-space Y and an arbitrary pointed G-space X. By Proposition 2.6.5 the category of G-spectra is generated under colimits and shifts by suspension G-spectra. The above map preserves colimits in the variable X, by compactness of Y. So, the map in Equation (35) is an equivalence, whenever Y is the suspension spectrum of a compact pointed space, and X is an arbitrary G-spectrum. Finally, me can fix X and pass to colimits in Y, to see that the map in Equation (35) is an equivalence for any two G-spectra X and Y.

By the universal property of G-spectra, see Proposition 2.6.10, the symmetric monoidal functor $\Phi^G \circ \mathrm{Infl}_1^G$ canonically identifies with the identity functor of spectra. □

The equivalence in Equation (35), specializes to an equivalence

$$(\widetilde{E\mathcal{P}_G} \otimes -)^G \to \Phi^G(-) \quad (36)$$

of symmetric monoidal functors $\mathrm{Sp}_G \to \mathrm{Sp}$, when we set $Y = \mathrm{Infl}_1^G(\mathbb{S})$.

Definition 2.6.25. For each compactly metrizable group G, we write $\mathcal{U}_G^\perp$ for the orthogonal complement of the G-fixed-points $(\mathcal{U}_G)^G$ in a complete G-universe G. For each finite dimensional orthogonal G-representation V, we write $S(V) \in \mathcal{S}_G$ for the Lie(G)-cofibrant G-space consisting of unit length vectors in V.

Note that any representation sphere $S^V \in \mathcal{S}_{G,*}$ identifies with cofiber of the based G-map $(S(V) \to *)_+$.

By [Sch18, Example 3.3.7.], for each $n \geq 0$, the unique G_n-map

$$\operatorname*{colim}_{V \subseteq \mathcal{U}_{G_n}^{\perp}} S(V) \to E\mathcal{P}_{G_n} \tag{37}$$

is an equivalence. We conclude from Equation (22) and Construction 2.5.16, that the unique G-map

$$\operatorname*{colim}_{V \subseteq \mathcal{U}_{G}^{\perp}} S(V) \to E\mathcal{P}_{G} \tag{38}$$

is an equivalence as well. Applying $\operatorname{cofib}((-)_+ \to *_+)$ to Equation (38) induces an equivalence

$$\operatorname*{colim}_{V \subseteq \mathcal{U}_{G}^{\perp}} S^V \to \widetilde{E\mathcal{P}_{G}} \tag{39}$$

under $S^0 \in \mathcal{S}_{G,*}$.

Remark 2.6.26. The previous discussion also works for a complex complete G-universe. Indeed, [Sch18, Example 3.3.7.] goes through in that setting.

Corollary 2.6.27. *The symmetric monoidal functor $\widetilde{E\mathcal{P}_G} \otimes (-) : \mathrm{Sp}_G \to \mathrm{Sp}_G$ is equivalent to the functor*

$$E \mapsto \operatorname*{colim}_{V \subseteq \mathcal{U}_{G}^{\perp}} S^V \otimes E$$

and the lax monoidal transformation $E^G \to \Phi^G(E)$ is equivalent to $E^G \to \operatorname*{colim}_{V \subseteq \mathcal{U}_{G}^{\perp}} (S^V \otimes E)^G$.

2.7. **Pro Global Homotopy Theory.**

2.7.1. *Unstable Pro Global Homotopy Theory.* Henriques and Gepner [HG07] defined the ∞-category $\mathcal{S}_{\mathrm{gl}}$ of *global spaces* as presheaf ∞-category on the global orbit category Glo. The objects of the global orbit category Glo are the global classifying spaces $\mathrm{B}_{\mathrm{gl}}G$ of compact Lie groups G. [LNP25, Proposition 6.3.] describes a functor out of the (homotopy coherent nerve of the topological) category of compact Lie groups

$$\mathrm{B}_{\mathrm{gl}}(-) : \mathrm{CptLie} \to \mathrm{Glo}, \qquad G \mapsto \mathrm{B}_{\mathrm{gl}}G\,,$$

which exhibits the set of homotopy classes of maps $[\mathrm{B}_{\mathrm{gl}}G, \mathrm{B}_{\mathrm{gl}}L]_{\mathrm{gl}}$ as the set of conjugacy classes of continuous group homomorphisms $G \to L$.

For every compact Lie group G, there is a *global homotopy orbit functor* $(-)_{/\!\!/ G} : \mathcal{S}_G \to \mathcal{S}_{\mathrm{gl}}$ sending an orbit $G/H \in \mathrm{Orb}_G$ to the (Yoneda image of the) global classifying space $\mathrm{B}_{\mathrm{gl}}H$. The *$G$-restriction functor* $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$ is defined as the right adjoint of the global homotopy orbit functor. It sends a global classifying space $\mathrm{B}_{\mathrm{gl}}L \in \mathrm{Glo}$ to the classifying space $\mathrm{B}_G(L) \in \mathcal{S}_G$ of G-equivariant L-principle bundles.

Construction 2.7.1. Let G be a compactly metrizable group. We construct a *G-restriction functor* $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$, which sends the global classifying space $\mathrm{B}_{\mathrm{gl}}L \in \mathrm{Glo}$ of a compact Lie group L to the classifying space $\mathrm{B}_G(L) \in \mathcal{S}_G$ of G-equivariant L-principle bundles. Inspired by the description of $\mathrm{B}_G(L)$ as colimit along inflations, see Equation (27), we want to define

$$\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G, \qquad \mathcal{X} \mapsto \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{Infl}_{G_n}^G (\mathrm{Res}_{G_n}(\mathcal{X}))\,,$$

where we chose an isomorphism from the topological group G to an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $\varprojlim_n G_n$ along surjective morphisms $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups G_n.

We construct the desired functor as the composite of the following four functors: Recall from [LNP25, Theorem 6.17.] that the G-restriction functors exhibit the category of global spaces as a partially lax limit

$$\mathcal{S}_{\mathrm{gl}} \to \operatorname*{laxlim^\dagger}_{\mathrm{B_{gl}}G \in \mathrm{Glo}^{\mathrm{op}}} \mathcal{S}_G, \qquad \mathcal{X} \mapsto \{\mathrm{Res}_G \mathcal{X}\}_{\mathrm{B_{gl}}G}$$

with respect to the marking of the global orbit category Glo by injective group homomorphisms. Employing the functoriality of lax limits in the indexing category, we obtain our second functor

$$\operatorname*{laxlim^\dagger}_{\mathrm{B_{gl}}G \in \mathrm{Glo}^{\mathrm{op}}} \mathcal{S}_G \to \operatorname*{laxlim}_{n\in\mathbb{N}} \mathcal{S}_{G_n}, \qquad \{X_G\}_{\mathrm{B_{gl}}G} \mapsto \{X_{G_n}\}_n$$

by restricting the partially lax limit along the opposite of the functor

$$\mathbb{N}^{\mathrm{op}} \to \mathrm{Glo}, \qquad n \mapsto \mathrm{B_{gl}}G_n.$$

We recall that the inflations exhibit $\mathcal{S}_G$ as the colimit of $n \mapsto \mathcal{S}_{G_n}$ in the ∞-category Pr^L. In particular, we obtain a natural transformation (valued in the ∞-category of large ∞-categories) from the functor $n \mapsto \mathcal{S}_{G_n}$ to the constant functor at $\mathcal{S}_G$. Employing the functoriality of lax limits in natural transformations, we obtain our third functor

$$\operatorname*{laxlim}_{n\in\mathbb{N}} \mathcal{S}_{G_n} \to \operatorname*{laxlim}_{n\in\mathbb{N}} \mathrm{const}(\mathcal{S}_G), \qquad (X_n)_n \mapsto (\mathrm{Infl}^G_{G_n}(X_n))_n.$$

Further, recall the equivalence of ∞-categories

$$\operatorname*{laxlim}_{n\in\mathbb{N}} \mathrm{const}(\mathcal{S}_G) \simeq \mathrm{Fun}(\mathbb{N}, \mathcal{S}_G)$$

from [Ber20]. Postcomposing the previous three functors with the colimit functor

$$\mathrm{colim}_n : \mathrm{Fun}(\mathbb{N}, \mathcal{S}_G) \to \mathcal{S}_G$$

yields the desired functor

$$\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G, \qquad \mathcal{X} \mapsto \mathrm{colim}_n \,\mathrm{Infl}^G_{G_n}\left(\mathrm{Res}_{G_n} \mathcal{X}\right).$$

Remark 2.7.2. Consider two *different* isomorphisms $G \cong \varprojlim_n G_n$ and $G \cong \varprojlim_m G'_m$ of topological groups with G_n, G_m compact Lie groups. Utilizing each of these isomorphisms in Construction 2.7.1 yields a priori different G-restriction functors $\mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$. Via a cofinality argument, one may produce a preferred natural equivalence between these. We refrain from discussing the higher coherence of this identification.

Observation 2.7.3. The restriction functor $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$ of Construction 2.7.1 preserves *finite* limits and all small colimits.

Proposition 2.7.4. *Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit with each G_n a compact Lie group. Let $n \in \mathbb{N}$ and consider a compact genuine G_n-space $Y \in \mathcal{S}^\omega_{G_n}$. Then, the inflations induce an equivalence*

$$\operatorname*{colim}_{k\geq n} \mathrm{Map}_{\mathcal{S}_{G_k}}(\mathrm{Infl}^{G_k}_{G_n}(Y), \mathrm{Res}_{G_k}\mathcal{X}) \to \mathrm{Map}_{\mathcal{S}_G}(\mathrm{Infl}^G_{G_n}(Y), \mathrm{Res}_G \mathcal{X}),$$

naturally in the global space $\mathcal{X} \in \mathcal{S}_{\mathrm{gl}}$.

Proof. We employ compactness of Y and the definition of $\mathrm{Res}_G(-)$ as a colimit:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{S}_G}(\mathrm{Infl}^G_{G_n}(Y), \mathrm{Res}_G \mathcal{X}) &\simeq \operatorname*{colim}_{k\geq n} \mathrm{Map}_{\mathcal{S}_G}(\mathrm{Infl}^G_{G_n}(Y), \mathrm{Infl}^G_{G_k}(\mathrm{Res}_{G_k}\mathcal{X})) \\
&\simeq \operatorname*{colim}_{k\geq n} \mathrm{Map}_{\mathcal{S}_G}(\mathrm{Infl}^G_{G_k}(\mathrm{Infl}^{G_k}_{G_n}(Y)), \mathrm{Infl}^G_{G_k}(\mathrm{Res}_{G_k}\mathcal{X})) \\
&\simeq \operatorname*{colim}_{k\geq n} \mathrm{Map}_{\mathcal{S}_{G_k}}(\mathrm{Infl}^G_{G_k}(Y), \mathrm{Res}_{G_k}\mathcal{X}),
\end{aligned}$$

where in the last step we used that the inflation functor $\mathrm{Infl}_{G_k}^G : \mathcal{S}_{G_k} \to \mathcal{S}_G$ is fully faithful by Construction 2.3.5. $\square$

Corollary 2.7.5. *Suppose surjective morphisms $\varphi_n : G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim G_n$ with each G_n a compact Lie group. For a subgroup $H \leq G$ with $H \in \mathrm{Lie}(G)$, we construct a preferred equivalence of spaces*

$$\operatorname*{colim}_{k\in\mathbb{N}} (\mathrm{Res}_{G_k} \mathcal{X})^{H_k} \xrightarrow{\simeq} (\mathrm{Res}_G \mathcal{X})^H$$

naturally in $\mathcal{X} \in \mathcal{S}_{\mathrm{gl}}$, where $H_k := \varphi_k(H) \leq G_k$

Proof. The fixed point functor is co-represented by the orbit $G/H \in \mathcal{S}_G$. We assumed that $H \in \mathrm{Lie}(G)$, so we may choose an $n \in \mathbb{N}$ such that the canonical map $G/H \to \mathrm{Infl}_{G_n}^G(G_n/H_n)$ is an equivalence. The result follows from Proposition 2.7.4 applied to $Y = G_n/H_n$. $\square$

Remark 2.7.6 (Global Classifying Space of Compactly Metrizable Groups)**.** Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit with each G_n a compact Lie group. The pro-extension of $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$

$$\mathrm{Res}_G : \mathrm{Pro}(\mathcal{S}_{\mathrm{gl}}) \to \mathcal{S}_G \tag{40}$$

has a left adjoint

$$(-)_{/\!\!/ G} : \mathcal{S}_G \to \mathrm{Pro}(\mathcal{S}_{\mathrm{gl}}). \tag{41}$$

by the dual of [Lur09, Proposition 5.3.5.13.]. For $H \leq G$ with $H \in \mathrm{Lie}(G)$, we obtain an equivalence

$$\begin{aligned}\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}\left((G/H)_{/\!\!/ G},\ \mathcal{X}\right) \simeq \mathrm{Map}_{\mathcal{S}_G}(G/H, \mathrm{Res}_G \mathcal{X}) &\overset{2.7.5}{\simeq} \mathrm{colim}_n\, \mathrm{Map}_{\mathcal{S}_{G_n}}(G_n/H_n, \mathrm{Res}_{G_n} \mathcal{X}) \\ &\simeq \mathrm{colim}_n\, \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}} H_n, \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}\left(\varprojlim_n \mathrm{B}_{\mathrm{gl}} H_n, \mathcal{X}\right)\end{aligned}$$

naturally in $\mathcal{X} \in \mathcal{S}_{\mathrm{gl}}$. Here, $H_n := \varphi_n(H) \leq G_n$ is the image of H under the projection $\varphi_n : G \to G_n$.

By the Yoneda Lemma, we obtain preferred equivalences

$$(G/H)_{/\!\!/ G} \simeq \varprojlim_n \mathrm{B}_{\mathrm{gl}} H_n, \qquad \text{and} \qquad (*)_{/\!\!/ G} \simeq \varprojlim_n \mathrm{B}_{\mathrm{gl}} G_n \tag{42}$$

in $\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})$. These inverse limits are *not* computed in the category of global spaces. As the *global homotopy orbit functor* $(-)_{/\!\!/ G} : \mathcal{S}_G \to \mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})$ preserves small colimits, it is uniquely determined by its restriction to the full subcategory of $\mathcal{S}_G$ spanned by orbits of subgroups $H \in \mathrm{Lie}(G)$.

The following Proposition is inspired by the special case of $\mathcal{X} = \mathrm{B}_{\mathrm{gl}} L$, which we saw in Section 2.5.2.

Proposition 2.7.7. *Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit with each G_n a compact Lie group. Let $\mathcal{X} \in \mathcal{S}_{\mathrm{gl}}$ be a global space. We abbreviate by $\mathcal{X}_{G_n} := \mathrm{Res}_{G_n}(\mathcal{X})$ and $\mathcal{X}_G := \mathrm{Res}_G(\mathcal{X})$ the restrictions of $\mathcal{X}$ in the sense of Construction 2.7.1. The functors*

$$(\mathcal{S}_{G_n})_{/\mathcal{X}_{G_n}} \xrightarrow{\mathrm{Infl}_{G_n}^G} (\mathcal{S}_G)_{/\,\mathrm{Infl}_{G_n}^G(\mathcal{X}_{G_n})} \to (\mathcal{S}_G)_{/\mathcal{X}_G}\,,$$

exhibit $(\mathcal{S}_G)_{/\mathcal{X}_G}$ as the colimit

$$\mathrm{colim}_n^L \left((\mathcal{S}_{G_n})_{/\mathcal{X}_{G_n}}\right) \xrightarrow{\simeq} (\mathcal{S}_G)_{/\mathcal{X}_G}$$

in the ∞-category Pr^L of presentable ∞-categories and left adjoint functors.

Proof. Taking compact objects $(-)^\omega$ is the inverse of Ind-completion $\mathrm{Ind}(-) : \mathrm{Cat}_\infty^{\mathrm{rex},\#} \to \mathrm{Pr}^{L,\omega}$, see [Lur09, 5.5.7.8.]. Because $\mathrm{Cat}_\infty^{\mathrm{rex},\#} \to \mathrm{Cat}_\infty$ preserves filtered colimits, we are reduced to showing that the inflations exhibit $(\mathcal{S}_G^\omega)_{/\mathcal{X}_G}$ as the colimit of $\left(\mathcal{S}_{G_n}^\omega\right)_{/\mathcal{X}_{G_n}}$ in Cat_∞. Recall that objects of filtered colimits in Cat_∞ are computed on underlying sets, see [BBB24, Section 2]. Because the map $\mathrm{colim}_n \mathrm{Infl}_{G_n}^G \mathcal{X}_{G_n} \to \mathcal{X}_G$ is an equivalence, the essential surjectivity of the functor

$$\mathrm{colim}_n \left(\left(\mathcal{S}_{G_n}^\omega\right)_{/\mathcal{X}_{G_n}} \right) \longrightarrow (\mathcal{S}_G^\omega)_{/\mathcal{X}_G}$$

follows from Equation (10). We will finish the proof by showing that this functor is fully-faithful: For this purpose we fix two objets $X, Y \in \left(\mathcal{S}_{G_n}^\omega\right)_{/\mathcal{X}_{G_n}}$ for some $n \in \mathbb{N}$. By the mapping space formula for filtered colimits in Cat_∞, see [BBB24, Lemma 2.3.], the inflations present the mapping space $\mathrm{Map}(X, Y)$ between the image of X and Y in

$$\mathrm{colim}_n \left(\left(\mathcal{S}_{G_n}^\omega\right)_{/\mathcal{X}_{G_n}} \right)$$

as the colimit

$$\operatorname*{colim}_{m \geq n} \mathrm{Map}_{/\mathcal{X}_{G_m}}(\mathrm{Infl}_{G_n}^{G_m}(X), \mathrm{Infl}_{G_n}^{G_m}(Y))$$

of the mapping spaces in the categories $\left(\mathcal{S}_{G_m}^\omega\right)_{/\mathcal{X}_{G_m}}$. Recall the fiber product formula for mapping spaces in any slice category:

$$\mathrm{Map}_{\mathcal{C}_{/x}}(z \xrightarrow{g} x, y \xrightarrow{f} x) \simeq \mathrm{fib}_g \left(\mathrm{Map}_{\mathcal{C}}(z, y) \xrightarrow{f_*} \mathrm{Map}_{\mathcal{C}}(z, x) \right)$$

Because filtered colimits commute with finite limits in spaces, we may write the mapping space $\mathrm{Map}(X, Y)$ as the fiber of the map

$$\operatorname*{colim}_{m \geq n} \mathrm{Map}_{\mathcal{S}_{G_m}}(\mathrm{Infl}_{G_n}^{G_m}(X), \mathrm{Infl}_{G_n}^{G_m}(Y)) \to \operatorname*{colim}_{m \geq n} \mathrm{Map}_{\mathcal{S}_{G_m}}(\mathrm{Infl}_{G_n}^{G_m}(X), \mathcal{X}_{G_m})$$

Because inflation is fully faithful, see Construction 2.3.5, the latter map of colimits identifies via the equivalence in Proposition 2.7.4 with the post-composition map

$$\mathrm{Map}_{\mathcal{S}_G}(\mathrm{Infl}_{G_n}^G(X), \mathrm{Infl}_{G_n}^G(Y)) \to \mathrm{Map}_{\mathcal{S}_G}(\mathrm{Infl}_{G_n}^G(X), \mathcal{X}_G)$$

along $\mathrm{Infl}_{G_n}^G(Y) \to \mathcal{X}_G$. By another application of the mapping space formula in slice categories, the fiber of the map under inspection computes the mapping space of the images of X and Y in the slice category $(\mathcal{S}_G)_{/\mathcal{X}_G}$. As this equivalence of mapping spaces was induced by the inflation $\mathrm{Infl}_{G_n}^G(-)$, the functor under inspection is fully faithful. □

2.7.2. *Stable Pro Global Homotopy Theory.*

Construction 2.7.8. Using the lax limit description of global spectra $\mathrm{Sp}_{\mathrm{gl}}$ from [LNP25, Theorem 11.10.], we mimic Construction 2.7.1 to define a symmetric monoidal functor

$$\mathrm{Res}_G : \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_G, \qquad \mathcal{X} \mapsto \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{Infl}_{G_n}^G \mathrm{Res}_{G_n}(\mathcal{X}),$$

whenever surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $G \cong \varprojlim G_n$ of compact Lie groups G_n. Here the arguments of Construction 2.7.1 go through verbatim in the stable setting, where only the symmetric monoidal structure wasn't discussed. Note that the canonical oplax symmetric monoidal structure of the colimit functor $\mathrm{Fun}(\mathbb{N}, \mathrm{Sp}_G) \to \mathrm{Sp}_G$ is strongly monoidal because $\mathbb{N}$ is sifted. Because [LNP25, Theorem 11.10] exhibits global spectra as partially lax limit in the $(\infty, 2)$-category of symmetric monoidal ∞-categories, our construction defines the functor Res_G as a composite of canonically symmetric monoidal functors.

As in the unstable setting, the restriction functor $\mathrm{Res}_G : \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_G$ of Construction 2.7.8 preserves small colimits.

Proposition 2.7.9. *Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit with each G_n a compact Lie group. Let $n \in \mathbb{N}$ and consider a compact genuine G_n-spectrum $Y \in \mathrm{Sp}^{\omega}_{G_n}$. Then, the inflations induce an equivalence*

$$\operatorname*{colim}_{m\geq n} \mathrm{map}_{\mathrm{Sp}_{G_m}}(\mathrm{Infl}^{G_m}_{G_n}(Y), \mathrm{Res}_{G_m}\mathcal{X}) \to \mathrm{map}_{\mathrm{Sp}_G}(\mathrm{Infl}^{G}_{G_n}(Y), \mathrm{Res}_G\mathcal{X}),$$

of mapping spectra naturally in the global spectrum $\mathcal{X} \in \mathrm{Sp}_{\mathrm{gl}}$.

Proof. We employ compactness of Y and the definition of $\mathrm{Res}_G(-)$ as a colimit:

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}_G}(\mathrm{Infl}^G_{G_n}(Y), \mathrm{Res}_G\mathcal{X}) &\simeq \operatorname*{colim}_{k\geq n} \mathrm{Map}_{\mathrm{Sp}_G}(\mathrm{Infl}^G_{G_n}(Y), \mathrm{Infl}^G_{G_k}(\mathrm{Res}_{G_k}\mathcal{X}))\\ &\overset{(2.6.2)}{\simeq} \operatorname*{colim}_{k\geq n}\operatorname*{colim}_{m\geq k} \mathrm{Map}_{\mathrm{Sp}_{G_m}}(\mathrm{Infl}^{G_m}_{G_n}(Y), \mathrm{Infl}^{G_m}_{G_k}(\mathrm{Res}_{G_k}\mathcal{X})).\end{aligned}$$

The result follows by cofinality. $\square$

Corollary 2.7.10. *Suppose surjective morphisms $\varphi_n : G \twoheadrightarrow G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim G_n$ with each G_n a compact Lie group. For a subgroup $H \leq G$ with $H \in \mathrm{Lie}(G)$, we construct a preferred equivalence of lax symmetric monoidal functors*

$$\mathrm{colim}_n\, (\mathrm{Res}_{G_n}(-))^{H_n} \xrightarrow{\simeq} (\mathrm{Res}_G(-))^H : \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp},$$

where $H_n := \varphi_n(H) \leq G_n$. By applying the homotopy group functor $\pi_* : \mathrm{Sp} \to \mathrm{Ab}^{\mathrm{gr}}$, we obtain a preferred (graded ring) isomorphism

$$\mathrm{colim}_n\, \pi^{H_n}_*(\mathrm{Res}_{G_n}\mathcal{X}) \to \pi^H_*(\mathrm{Res}_G(\mathcal{X})). \tag{43}$$

naturally in the (homotopy ring) global spectrum $\mathcal{X} \in \mathrm{Sp}_{\mathrm{gl}}$.

Proof. The fixed point functor is corepresented by the suspension $\Sigma^\infty_+(G/H)$ of the orbit $G/H \in \mathcal{S}_G$. By assumption $H \in \mathrm{Lie}(G)$, so that we may choose an $n \in \mathbb{N}$ such that the canonical map $G/H \to \mathrm{Infl}^G_{G_n}(G_n/H_n)$ is an equivalence. The result follows from Proposition 2.7.9 applied to the orbit $Y = \Sigma^\infty_+ G_n/H_n \in \mathrm{Sp}^\omega_{G_n}$. Note that the natural transformation in Proposition 2.7.9 is lax symmetric monoidal in the variable $\mathcal{X}$ whenever Y is a co-commutative co-algebra object. $\square$

2.8. **Equivariant Thom Spectra.**

2.8.1. *Representability of Equivariant K-Theory.*

Construction 2.8.1. Consider the monoid $\mathrm{B}\mathbf{O}_{\mathrm{gl}} := \coprod_{d\in\mathbb{N}} \mathrm{B}_{\mathrm{gl}}(O(d)) \in \mathrm{CMon}(\mathcal{S}_{\mathrm{gl}})$ and its group completion[7] $\mathbf{BOP}_{\mathrm{gl}} \in \mathrm{CGrp}(\mathcal{S}_{\mathrm{gl}})$. For any compactly metrizable group G, the G-restriction functor $\mathrm{Res}_G : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$ preserves finite products and small colimits. In particular, the G-restriction functor preserves group completions. Specializing to our example, the G-restriction of a group completion $\mathrm{B}\mathbf{O}_{\mathrm{gl}} \to \mathbf{BOP}_{\mathrm{gl}}$ is a group completion

$$\mathrm{B}\mathbf{O}_G \to \mathbf{BOP}_G := \mathrm{Res}_G(\mathbf{BOP}_{\mathrm{gl}})$$

of the monoid

$$\mathrm{B}\mathbf{O}_G := \mathrm{Res}_G(\mathrm{B}\mathbf{O}_{\mathrm{gl}}) \simeq \coprod_{d\in\mathbb{N}} \mathrm{B}_G(O(d)) \in \mathrm{CMon}(\mathcal{S}_G).$$

[7]In our convention a bold $\boldsymbol{B}$ stands for the group completed version.

Let X be a $\mathrm{Lie}(G)$-cofibrant G-space. By Proposition 2.5.12, pullback along the universal G-equivariant vector bundles $\gamma_d := \mathrm{E}_G O(d) \times_{O(d)} \mathbb{R}^d$ defines a monoid isomorphism

$$[X, \mathrm{B}\mathbf{O}_G]^G \to \mathrm{vect}^0_G(X) \tag{44}$$

between the set of G-equivariant homotopy classes of maps and the set isomorphism classes of G-equivariant vector bundles over X. We define $\mathbf{KO}^0_G(X)$ as the Grothendieck group of the monoid $\mathrm{vect}^0_G(X)$. By its universal property, we extend the above monoid isomorphism to a homomorphism

$$\mathbf{KO}^0_G(X) \to [X, \mathbf{BOP}_G]^G$$

to the group of G-equivariant homotopy classes of maps $X \to \mathbf{BOP}_G$.

Theorem 2.8.2. *For every compactly metrizable group G and every compact* $\mathrm{Lie}(G)$*-cofibrant topological G-space X, the group homomorphism from Construction 2.8.1*

$$\mathbf{KO}^0_G(X) \to [X, \mathbf{BOP}_G]^G$$

is an isomorphism from the Grothendieck group of isomorphism classes of G-equivariant vector bundles on X to the monoid of G-equivariant homotopy classes of maps $X \to \mathbf{BOP}_G$.

Proof. In the case that G is a compact Lie group, this is a result of Schwede's, see [Sch18, Theorem 2.4.10.]. We choose surjective morphisms $G \to G_n$ exhibiting the topological group G as $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim_n G_n$ of compact Lie groups G_n. Because X is $\mathrm{Lie}(G)$-cofibrant and compact, the G action on X factors through some G_n, i.e. we may choose an object $\widetilde{X} \in \mathcal{S}^{\omega}_{G_n}$ so that the inflated object $\mathrm{Infl}^G_{G_n}(\widetilde{X}) \in \mathcal{S}^{\omega}_G$ represents the genuine G-space X. By Proposition 2.7.4, the inflations induce a bijection

$$\operatorname*{colim}_{m \geq n} \left[\mathrm{Infl}^{G_m}_{G_n}(\widetilde{X}), \mathbf{BOP}_{G_m}\right]^{G_m} \to [X, \mathbf{BOP}_G]^G .$$

As we already know the result for compact Lie groups, we are reduced to checking that the group homomorphism

$$\operatorname*{colim}_{m \geq n} \mathbf{KO}^0_{G_m}(\mathrm{Infl}^{G_m}_{G_n}(\widetilde{X})) \to \mathbf{KO}^0_G(X)$$

is an isomorphism, as well. Because group completion is a left adjoint, it suffices to show that the monoid map

$$\operatorname*{colim}_{m \geq n} \mathrm{vect}^0_{G_m}(\mathrm{Infl}^{G_m}_{G_n}(\widetilde{X})) \to \mathrm{vect}^0_G(X)$$

is a bijection. By the representability result of Equation (44), it suffices to show that the inflations induce a bijection

$$\operatorname*{colim}_{m \geq n} \left[\mathrm{Infl}^{G_m}_{G_n}(\widetilde{X}), \mathrm{B}\mathbf{O}_{G_m}\right]^{G_m} \to [X, \mathrm{B}\mathbf{O}_G]^G .$$

This follows from another application of Proposition 2.7.4. □

2.8.2. *Equivariant Thom Spectra for Compact Lie Groups.*

Remark 2.8.3. Let G be a compactly metrizable group. We believe that the most elegant and conceptional construction of the G-equivariant complex bordism spectrum $\mathbf{MU}_G$ is via the equivariant symmetric monoidal Thom spectrum functor. Our construction of this Thom spectrum functor will be based on forthcoming work by Emma Brink. In Equation (47) and Corollary 2.9.18, we provide two descriptions of $\mathbf{MU}_G$: a "global" description and a "telescoping" description, respectively. Either of these can be taken as an ad hoc definition of the ring G-spectrum $\mathbf{MU}_G$, and both are independent of the existence of a Thom spectrum functor—in particular, they do not depend on Brink's upcoming results.

Furthermore, outside of this section on Equivariant Thom Spectra, we refer only to the "global" or "telescoping" description of $\mathbf{MU}_G$, and make no use of the symmetric monoidal Thom spectrum functor. Consequently, all of our results outside this section are independent of Brink's work, provided we establish that the "global" and "telescoping" descriptions of $\mathbf{MU}_G$ agree as homotopy ring G-spectra. We provide an alternative proof of this agreement—one that does not rely on the Thom spectrum functor—in Remark 2.9.20.

Construction 2.8.4 (Brink)**.** Upcoming work of Emma Brink will include the construction of a functor

$$\mathrm{Th}_\bullet : \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L))$$

together with the following two equivalences:

(1) An equivalence from the composite

$$\mathrm{Glo}^{\mathrm{op}} \xrightarrow{\mathrm{Th}_\bullet} \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L)) \xrightarrow{\text{target}} \mathrm{CAlg}(\mathrm{Pr}^L)$$

to the functor $\mathrm{Sp}_\bullet : \mathrm{Glo}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$, $\mathrm{B}_{\mathrm{gl}}G \mapsto \mathrm{Sp}_G$ from [LNP25].

(2) Recall that Orb is a wide subcategory of Glo with $\mathrm{Orb}_{/\mathrm{B}_{\mathrm{gl}}G} = \mathrm{Orb}_G$ for all $G \in \mathrm{CptLie}$. Brink constructs an equivalence from the functor

$$\mathrm{Glo}^{\mathrm{op}} \xrightarrow{\mathrm{Th}_\bullet} \mathrm{Fun}([1], \mathrm{Pr}^L) \xrightarrow{\text{source}} \mathrm{Pr}^L$$

to the composite functor

$$(\mathcal{S}_\bullet)_{/\mathbf{BOP}_{\mathrm{gl}}} :\ \mathrm{Glo}^{\mathrm{op}} \xrightarrow{\left(\mathrm{Orb}_{/-} \times_{\mathrm{Glo}} \mathrm{Glo}_{/\mathbf{BOP}_{\mathrm{gl}}}\right)^{\mathrm{op}}} \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathcal{S})} \mathrm{Pr}^L.$$

(3) Moreover, Brink shows that Elemendorf's equivalence $\mathcal{P}(\mathrm{Orb}_G) \simeq \mathcal{S}_G$ induces a symmetric monoidal equivalence from the source of the G-equivariant Thom spectrum functor

$$\mathrm{Th}_G := \mathrm{Th}_{\mathrm{B}_{\mathrm{gl}}G} :\ \mathcal{P}\left(\mathrm{Orb}_G \times_{\mathrm{Glo}} \mathrm{Glo}_{/\mathbf{BOP}_{\mathrm{gl}}}\right) \to \mathrm{Sp}_G,$$

to the slice category

$$\mathcal{P}\left(\mathrm{Orb}_G \times_{\mathrm{Glo}} \mathrm{Glo}_{/\mathbf{BOP}_{\mathrm{gl}}}\right) \xrightarrow{\simeq} (\mathcal{S}_G)_{/\mathbf{BOP}_G},$$

which carries the slice monoidal structure[8] induced by the monoid structure of $\mathbf{BOP}_G$ and the cartesian monoidal structure on $\mathcal{S}_G$.

Remark 2.8.5. For $\alpha : G \to K$ a morphism of compact Lie groups, the functor

$$(\mathcal{S}_\alpha)_{/\mathbf{BOP}_{\mathrm{gl}}} : (\mathcal{S}_K)_{/\mathbf{BOP}_K} \to (\mathcal{S}_G)_{/\mathbf{BOP}_G}$$

sends $(f : X \to \mathbf{BOP}_K)$ to the composite

$$\alpha^*(X) \xrightarrow{\alpha^*(f)} \alpha^*(\mathbf{BOP}_K) \xrightarrow{\mathrm{Res}_\alpha} \mathbf{BOP}_G,$$

where the map Res_α is obtained by evaluating the lax natural transformation $\mathrm{Res}_\bullet : \mathrm{const}(\mathcal{S}_{\mathrm{gl}}) \Rightarrow \mathcal{S}_\bullet$ from [LNP25, Theorem 6.7.] at the morphism $\alpha : G \to K$. We conclude by the naturality of Res_α, that the map $\alpha^*(\mathbf{BOP}_K) \xrightarrow{\mathrm{Res}_\alpha} \mathbf{BOP}_G$ is the group completion of the canonical monoid map $\alpha^* \mathrm{B}\mathbf{O}_K \to \mathrm{B}\mathbf{O}_G$, which, in turn, is induced by the classifying maps of the G-restrictions $\alpha^*\gamma_d$ of the universal K-equivariant vector bundles.

Theorem 2.8.6 (Brink)**.** *For any compact Lie-group G, the Thom spectrum functor*

$$\mathrm{Th}_G :\ (\mathcal{S}_G)_{/\mathbf{BOP}_G} \to \mathrm{Sp}_G$$

obtained by evaluating $\mathrm{Th}_\bullet : \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L))$ at the global classifying space $\mathrm{B}_{\mathrm{gl}}G$ satisfies the following properties

[8] See [Lur17, Remark 2.2.2.5.] for a construction of the slice monoidal structure.

- *For any closed subgroup $H \leq G$, the Beck-Chevalley transformation of the equivalence* $\mathrm{Th}_H \circ \mathrm{Res}^G_H \simeq \mathrm{Res}^G_H \circ \mathrm{Th}_G$ *is an equivalence* $\mathrm{Ind}^G_H \circ \mathrm{Th}_H \xRightarrow{\simeq} \mathrm{Th}_G \circ \mathrm{Ind}^G_H$.
- *The composite*

$$(\mathcal{S}_G)_{/\,\mathrm{B}_G(O(n))} \to (\mathcal{S}_G)_{/\,\mathbf{BOP}_G} \xrightarrow{\mathrm{Th}_G(-)} \mathrm{Sp}_G$$

factors as

$$(\mathcal{S}_G)_{/\,\mathrm{B}_G(O(n))} \to \mathcal{S}_{G,*} \xrightarrow{\Sigma^\infty} \mathrm{Sp}_G,$$

where the first functor is induced by the point-set model Thom space construction

$$(\mathrm{Top}_G)_{/\,\mathrm{Gr}_n(\mathcal{U}_G)} \to \mathrm{Top}_{G,*}, \qquad \xi \mapsto X^\xi,$$

which preserves colimits, as well as cofibrations and weak equivalences for the slice genuine model structure.

2.8.3. *Equivariant Thom Spectra for Compactly Metrizable Groups.*

Construction 2.8.7. We Ind(−) extend the composite functor

$$\mathrm{CptLie}^{\mathrm{op}} \to \mathrm{Glo}^{\mathrm{op}} \xrightarrow{\mathrm{Th}_\bullet} \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L))$$

to $\mathrm{Pro}(\mathrm{CptLie})^{\mathrm{op}}$ and restrict along the pro-analogue functor from Definition 2.2.2 to get the *Thom spectrum functor*

$$\mathrm{Th}_\bullet : \mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \to \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L)).$$

Observation 2.8.8. The composite functor

$$\mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \xrightarrow{\mathrm{Th}_\bullet} \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L)) \xrightarrow{\mathrm{target}} \mathrm{CAlg}(\mathrm{Pr}^L)$$

is equivalent to the functor $\mathrm{Sp}_\bullet : \mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ of Construction 2.2.3 via Ind extending the equivalence from the Lie group case (1). The composite functor

$$\mathrm{Th}_\bullet : \mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \xrightarrow{\mathrm{Th}_\bullet} \mathrm{Fun}([1], \mathrm{CAlg}(\mathrm{Pr}^L)) \xrightarrow{\mathrm{source}} \mathrm{CAlg}(\mathrm{Pr}^L)$$

is equivalent to the functor

$$\mathrm{Grp}(\mathrm{CptMet})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L), \qquad G \mapsto (\mathcal{S}_G)_{/\,\mathbf{BOP}_G}$$

via Proposition 2.7.7.

Definition 2.8.9. Let G be a compactly metrizable group. We define the *G-equivariant Thom spectrum functor*

$$\mathrm{Th}_G : (\mathcal{S}_G)_{/\,\mathbf{BOP}_G} \longrightarrow \mathrm{Sp}_G$$

as the evaluation of the Thom spectrum functor $\mathrm{Th}_\bullet$ from Construction 2.8.7 at $G \in \mathrm{Grp}(\mathrm{CptMet})$.

By its construction, the Thom spectrum functor is symmetric monoidal.

Remark 2.8.10. The functoriality of the Thom spectrum functor of Construction 2.8.7 encodes a commutative square

$$\begin{array}{ccc}
(\mathcal{S}_K)_{/\,\mathbf{BOP}_K} & \xrightarrow{\mathrm{Th}_K} & \mathrm{Sp}_K \\
\downarrow{\scriptstyle \alpha^*} & & \\
(\mathcal{S}_G)_{/\alpha^*\,\mathbf{BOP}_K} & & \downarrow{\scriptstyle \alpha^*} \\
\downarrow & & \\
(\mathcal{S}_G)_{/\,\mathbf{BOP}_G} & \xrightarrow{\mathrm{Th}_G} & \mathrm{Sp}_G
\end{array}$$

in $\mathrm{CAlg}(\mathrm{Pr}^L)$ for any continuous homomorphism $\alpha : G \to K$ of compactly metrizable groups, where $\alpha^* \mathbf{BOP}_K \to \mathbf{BOP}_G$ is the group completion of the canonical monoid map $\alpha^* \mathrm{B}\mathbf{O}_K \to \mathrm{B}\mathbf{O}_G$, which, in turn, is induced by the classifying maps of the restrictions $\alpha^*\gamma_d$ of the universal K-equivariant vector bundles.

We have seen in Example 2.5.22 that a G-equivariant vector bundle on a $\mathrm{Lie}(G)$-cofibrant G-space X is classified by a G-map to the $\mathrm{Lie}(G)$-cofibrant G-space $\mathrm{Gr}_n(\mathcal{U}_G)$.

Theorem 2.8.11. *For any compactly metrizable group G, the Thom spectrum functor*

$$\mathrm{Th}_G : (\mathcal{S}_G)_{/\,\mathbf{BOP}_G} \longrightarrow \mathrm{Sp}_G$$

satisfies the following properties:

(1) The G-equivariant Thom spectrum functor Th_G preserves small colimits.

(2) For any subgroup $H \leq G$ with $H \in \mathrm{Lie}(G)$, the Beck-Chevalley transformation of the natural equivalence $\mathrm{Th}_H \circ \mathrm{Res}^G_H \simeq \mathrm{Res}^G_H \circ \mathrm{Th}_G$ is a natural equivalence

$$\mathrm{Ind}^G_H \circ \mathrm{Th}_H \overset{\simeq}{\Longrightarrow} \mathrm{Th}_G \circ \mathrm{Ind}^G_H\,.$$

(3) For every $n \in \mathbb{N}$, the composite

$$(\mathcal{S}_G)_{/\,\mathrm{B}_G(O(n))} \to (\mathcal{S}_G)_{/\,\mathbf{BOP}_G} \xrightarrow{\mathrm{Th}_G(-)} \mathrm{Sp}_G$$

factors as

$$(\mathcal{S}_G)_{/\,\mathrm{B}_G(O(n))} \to \mathcal{S}_{G,*} \xrightarrow{\Sigma^\infty} \mathrm{Sp}_G,$$

where the first functor is induced by the point-set model Thom space construction[9]

$$(\mathrm{Top}_G)_{/\,\mathrm{Gr}_n(\mathcal{U}_G)} \to \mathrm{Top}_{G,*}, \qquad \xi \mapsto X^\xi,$$

which preserves small colimits, as well as, cofibrations and weak equivalences, for the slice $\mathrm{Lie}(G)$-projective model structure.

Proof. The first statement is clear. For the second statement, we may reduce to the case of compact objects, employing the preservation of small colimits. The result follows from the compact Lie group case, see Theorem 2.8.6(i), via the compatibility (2.8.10) of equivariant Thom spectra with inflations. Here, we may commute the relevant inflation and induction, by Lemma 2.6.19. Concerning the third statement, we make the analogous point-set model construction of the Thom space functor, as in the compact Lie group case of Theorem 2.8.6 (ii). The same proof as in the compact Lie group case shows that the point-set Thom space construction preserves colimits and weak equivalences. Because any generating $\mathrm{Lie}(G)$-cofibration is inflated from a compact Lie group, our point set Thom space construction preserves $\mathrm{Lie}(G)$-cofibrations, since this is true in the compact Lie group case. Now that we constructed all functors at hand, we produce the required factorization from the compact Lie group case, by the passage to the colimit along inflations, using Proposition 2.7.7. □

[9]For a G-equivariant vector bundle over a compact $\mathrm{Lie}(G)$-cofibrant space, the point-set Thom space construction is given by the one-point compactification of the total space. In general, it is the (homotopy cofiber) of the sphere bundle.

2.9. Examples of Pro Compact Lie Spectra.

Construction 2.9.1 (Borel Spectra)**.** For any compact Lie group G, the *Borel construction* is right adjoint to the homotopy orbits functor $(-)_{hG} : \mathrm{Sp}_G \to \mathrm{Sp}$. Recall that the restriction to the trivial group $\mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}$ admits a right adjoint $(-)^b : \mathrm{Sp} \to \mathrm{Sp}_{\mathrm{gl}}$ such that the Borel construction factors as the composite

$$\mathrm{Sp} \xrightarrow{(-)^b} \mathrm{Sp}_{\mathrm{gl}} \xrightarrow{\mathrm{Res}_G} \mathrm{Sp}_G\,.$$

If surjective morphisms $G \to G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed limit $G \cong \varprojlim_n G_n$ of compact Lie groups G, we call the lax symmetric monoidal composite functor

$$(-)^{bG} :\ \mathrm{Sp} \xrightarrow{(-)^b} \mathrm{Sp}_{\mathrm{gl}} \xrightarrow{\mathrm{Res}_G} \mathrm{Sp}_G,$$

the *Borel construction*. The Borel construction $(-)^{bG}$ preserves finite colimits, but, admits no adjoints, in general. For a (homotopy ring) spectrum $R \in \mathrm{Sp}$, Equation (43) describes a (graded ring) isomorphism

$$\mathrm{colim}_n\, R^{-*}(BG_n) \to \pi_*^G(R^{bG}) \tag{45}$$

Construction 2.9.2. Analogously to Construction 2.8.1, for any compactly metrizable group G, we construct the grouplike monoid $\mathbf{BUP}_G \in \mathrm{CMon}(\mathcal{S}_G)$ by group completing the monoid $\mathrm{B}\mathbf{U}_G := \coprod_{d\in\mathbb{N}} \mathrm{B}_G(U(d))$ in G-spaces.

Construction 2.9.3 (Equivariant Complex K-Theory)**.** For a compactly metrizable group G, the *G-equivariant periodic complex K-theory spectrum* $\mathbf{KU}_G \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is defined as the G-restriction of the periodic global complex K-theory spectrum $\mathbf{KU} \in \mathrm{CAlg}(\mathrm{Sp}_{\mathrm{gl}})$ of [Sch18, Section 6.3.]. The equivalence $\mathbf{BUP}_{\mathrm{gl}} \to \Omega^\infty_{\mathrm{gl}}\mathbf{KU}$ in $\mathrm{CMon}(\mathcal{S}_{\mathrm{gl}})$ induces an equivalence

$$\mathbf{BUP}_G \to \Omega^\infty \mathbf{KU}_G$$

in $\mathrm{CMon}(\mathcal{S}_G)$ on G-restrictions. By the unitary analog of Theorem 2.8.2, for any compact $\mathrm{Lie}(G)$-cofibrant topological G-space X, there is a preferred isomorphism from the unreduced zeroth $\mathbf{KU}$-cohomology group $\mathbf{KU}^0_G(X_+)$ to the Grothendieck group of isomorphism classes of G-equivariant complex vector bundles on X: $\mathbf{KU}^0_G(X_+) \cong (\mathrm{vect}^{\mathbb{C}}_G(X))^{\mathrm{grp}}$.

Construction 2.9.4 (Symmetric Monoidal Thom Spectra)**.** Let G be a compactly metrizable group. Let $\xi : X \to \mathbf{BOP}_G$ be a morphism in the category $\mathrm{CMon}(\mathcal{S}_G)$ of commutative monoids in G-spaces. We define the *Thom spectrum* of ξ by applying the symmetric monoidal Thom spectrum functor $\mathrm{Th}_G : (\mathcal{S}_G)_{/\mathbf{BOP}_G} \to \mathrm{Sp}_G$ to $\xi \in \mathrm{CAlg}((\mathcal{S}_G)_{/\mathbf{BOP}_G})$.
Here we use that the forgetful functor induces an equivalence $\mathrm{CAlg}((\mathcal{S}_G)_{/\mathbf{BOP}_G}) \simeq \mathrm{CMon}(\mathcal{S}_G)_{/\mathbf{BOP}_G}$. Indeed this is clear from the construction of the slice monoidal structure on $(\mathcal{S}_G)_{/\mathbf{BOP}_G}$, see [Lur17, Remark 2.2.2.5.].

In the following we focus on unitary groups, but the same arguments and definitions apply verbatim for the family of orthogonal groups or symplectic groups.

Construction 2.9.5 (Equivariant Complex Bordism Spectrum)**.** Let G be a compactly metrizable group.

- The *Thom spectrum of the positive additive complex Grassmannian*

$$\mathrm{B}\mathbf{U}^\gamma_G \simeq \bigoplus_{d\in\mathbb{N}} \Sigma^\infty\, \mathrm{B}_G(U(d))^{\gamma_d} \in \mathrm{CAlg}(\mathrm{Sp}_G)$$

  is defined as the Thom spectrum of the composite

$$\mathrm{B}\mathbf{U}_G \to \mathrm{B}\mathbf{O}_G \to \mathbf{BOP}_G\,.$$

This map $\mathrm{B}\mathbf{U}_G \to \mathbf{BOP}_G$ factors uniquely through the group completion:

$$\mathrm{B}\mathbf{U}_G \to \mathbf{BUP}_G \to \mathbf{BOP}_G\,.$$

The symmetric monoidal Thom spectrum of $\mathbf{BUP}_G \to \mathbf{BOP}_G$ is the *periodic complex bordism spectrum*

$$\mathbf{MUP}_G \in \mathrm{CAlg}(\mathrm{Sp}_G).$$

- The *complex dimension map*

$$\dim : \mathbf{BUP}_G \to \mathbb{Z}, \qquad \xi \mapsto \dim(\xi)$$

is defined as the group completion of the canonical morphism

$$\mathrm{B}\mathbf{U}_G^{\gamma} := \coprod_{d\in\mathbb{N}} \mathrm{B}_G(U(d)) \to \mathbb{N}$$

in $\mathrm{CMon}(\mathrm{Sp}_G)$.

- The *G-equivariant complex bordism spectrum* $\mathbf{MU}_G \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is defined as the Thom spectrum of the composite

$$\mathbf{BU}^0_G \to \mathbf{BUP}_G \to \mathbf{BOP}_G,$$

where $\mathbf{BU}^0_G$ is defined as the fiber over $0 \in \mathbb{Z}$ under the complex dimension map. The dimension map induces an equivalence $\coprod_{d\in\mathbb{Z}} \mathbf{BU}^0_G \to \mathbf{BUP}_G$ on underlying G-spaces, from which we obtain a preferred equivalence

$$\bigoplus_{d\in\mathbb{Z}} \mathbf{MU}_G \otimes \mathbb{S}^{2d} \xrightarrow{\simeq} \mathbf{MUP}_G \tag{46}$$

of G-spectra.

Construction 2.9.6. Suppose morphisms $G \to G_n$ exhibit a topological group G as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $G \cong \varprojlim_n G_n$ along surjective continuous homomorphism $G_{n-1} \twoheadleftarrow G_n$ of compact Lie groups. We apply the G-equivariant Thom spectrum functor to the canonical equivalence

$$\mathrm{colim}_n\, \mathrm{Infl}^G_{G_n}(\mathbf{BU}^0_{G_n}) \to \mathbf{BU}^0_G$$

of commutative monoid objects, to obtain a preferred equivalence

$$\mathrm{colim}_n\, \mathrm{Infl}^G_{G_n}(\mathbf{MU}_{G_n}) \to \mathbf{MU}_G \tag{47}$$

in $\mathrm{CAlg}(\mathrm{Sp}_G)$. The analogous constructions work for the spectra $\mathbf{MUP}_G$ and $\mathrm{B}\mathbf{U}^{\gamma}_G$. In particular, by (the proof of) Equation (43), the inflations induce a preferred graded ring isomorphism

$$\mathrm{colim}_n\, \pi^{G_n}_*(\mathbf{MU}_{G_n}) \to \pi^G_*(\mathbf{MU}_G). \tag{48}$$

Remark 2.9.7 (Global Structure of Thom spectra)**.** The functoriality of the equivariant Thom spectrum functor $\mathrm{Th}_\bullet : \mathrm{Glo}^{\mathrm{op}} \times [1] \to \mathrm{CAlg}(\mathrm{Pr}^L)$ and the lax limit description of the categories $\mathcal{S}_{\mathrm{gl}}$ and $\mathrm{Sp}_{\mathrm{gl}}$ allows us to write down a symmetric monoidal Thom spectrum functor

$$\mathrm{Th}_{\mathrm{gl}} : (\mathcal{S}_{\mathrm{gl}})_{/\mathbf{BOP}_{\mathrm{gl}}} \to \mathrm{Sp}_{\mathrm{gl}}$$

that restricts to the G-equivariant Thom spectrum functor for any compact Lie group G. Mimicking the G-equivariant construction, we can define global bordism spectra[10] such as

$$\mathbf{MUP} := \mathrm{Th}_{\mathrm{gl}}(\mathbf{BUP}_{\mathrm{gl}} \to \mathbf{BOP}_{\mathrm{gl}}) \in \mathrm{CAlg}(\mathrm{Sp}_{\mathrm{gl}}).$$

[10] Upcoming work of Schwede's will identify the orthogonal spectrum model [Sch18, Example 6.1.53.] of the global periodic complex bordism spectrum $\mathbf{MUP} \in \mathrm{CAlg}(\mathrm{Sp}_{\mathrm{gl}})$ with the above global Thom spectrum construction. Independent of Schwede's upcoming work, it follows from the argument in Remark 2.9.20 that the G-restriction of Schwede's global periodic bordism spectrum from [Sch18, Example 6.1.53.] agrees with our construction of $\mathbf{MUP}_G$, as homotopy ring G-spectrum. The same applies for the (non-periodic) version $\mathbf{MU}_G$.

Now, Construction 2.9.6 describes preferred equivalences

$$\mathrm{Res}_G(\mathbf{MUP}) \xrightarrow{\simeq} \mathbf{MUP}_G, \qquad \mathrm{Res}_G(\mathbf{MU}) \xrightarrow{\simeq} \mathbf{MU}_G \qquad \text{and} \qquad \mathrm{Res}_G(\mathrm{B}\mathbf{U}^{\gamma}_{\mathrm{gl}}) \xrightarrow{\simeq} \mathrm{B}\mathbf{U}^{\gamma}_G$$

in $\mathrm{CAlg}(\mathrm{Sp}_G)$, for any compactly metrizable group G. Under this identification, Equation (48) becomes an honest special case of our computation (43) of the homotopy groups of G-restricted global spectra.

2.9.1. *Telescoping Periodic Complex Bordism Spectrum.*

Notation 2.9.8. Let G be a compactly metrizable group. For G-spaces X and Y, we abbreviate the set of homotopy classes of G-maps by

$$[X, Y]^G := \pi_0 \operatorname{Map}_{\mathcal{S}_G}(X, Y).$$

We use similar notation for maps of G-spectra and suppress the notation of Σ^∞, when the context is clear.

Notation 2.9.9 (RO(G)-grading)**.** Let G be a compactly metrizable group, let $H \le G$ be a subgroup and let $V : * \to \mathbf{BOP}_H$ represent a virtual H-representation $V \in \mathrm{RO}(H)$. Consistently with Definition 2.6.9, we call $\mathbb{S}^V := \mathrm{Th}(V) \in \mathrm{Sp}_H$ the *representation sphere* of V. For any G-spectrum X, we denote by

$$\pi^H_V(X) := [\mathbb{S}^V, \mathrm{Res}^G_H(X)]^H$$

the *representation graded homotopy group* of X.

Lemma 2.9.10. Let G be a compactly metrizable group. Let $f : \mathrm{B}\mathbf{U}_G \to X$ be a morphism in $\mathrm{CMon}(\mathcal{S}_G)$. The following are equivalent:

(a) The map f is invertible in the monoid $[\mathrm{B}\mathbf{U}_G, X]^G$.
(b) For any $d \in \mathbb{N}$ and any d-dimensional unitary G-representation V, the composite G-map

$$* \xrightarrow{V} \mathrm{B}_G(U(d)) \to \mathrm{B}\mathbf{U}_G \xrightarrow{f} X$$

represents an invertible object of the monoid $\pi^G_0(X) := [*, X]^G$.

Proof. Because restriction defines a group homomorphism $[\mathrm{B}\mathbf{U}_G, X]^G \to [*, X]^G$, a) implies b). For the converse, assuming b), it suffices to prove that the shear map

$$\chi(f) : \mathrm{B}\mathbf{U}_G \times X \to \mathrm{B}\mathbf{U}_G \times X, \qquad (y, x) \mapsto (y, f(y) + x)$$

is an equivalence of G-spaces. Recall that the fix point functors $(-)^H : \mathcal{S}_G \to \mathcal{S}$, for $H \le G$ running through all $H \in \mathrm{Lie}(G)$ are jointly conservative.

We claim that f^H factors through the inclusion $(X^H)^\times \to X^H$ of invertible components of X^H: Any map $W \in \pi^H_0(\mathrm{B}\mathbf{U}_G)$ represents a unitary H-representation W. We choose a H-linear isometric embedding $W \hookrightarrow \mathrm{Res}^G_H(V)$ for a finite dimensional unitary G-representation V. By assumption, $f_*(V) \in \pi^G_0(X)$ is invertible. Consequently, the restriction $f_*(\mathrm{Res}^G_H(V)) \in \pi^H_0(X)$ is invertible, as well. Consider the orthogonal decomposition $\mathrm{Res}^G_H(V) = W \oplus W^\perp$, as an unitary H-representation. In $\pi^H_0(\mathrm{B}\mathbf{U}_G)$ we have $\mathrm{Res}^G_H(V) = W + W^\perp$. Moreover, f is a monoid map, so, the equation $f_*(W) + f_*(W^\perp) = f_*(\mathrm{Res}^G_H(V))$ holds in $\pi^H_0(X)$. We conclude, that $f_*(W)$ and $f_*(W^\perp)$ are both invertible in $\pi^H_0(X)$.

This proves the claim because $\pi^H_0(X) := \pi_0(X^H)$. By construction, $(X^H)^\times$ is a grouplike $\mathbb{E}_\infty$-space. Thus, f^H admits an inverse in $[\mathrm{B}\mathbf{U}^H_G, X^H]$. This implies that the shear map

$$\chi(f^H) : \mathrm{B}\mathbf{U}^H_G \times X^H \to \mathrm{B}\mathbf{U}^H_G \times X^H, \qquad (y, x) \mapsto (y, f^H(y) + x)$$

is an equivalence. But, the fixed point functor $(-)^H$ preserves products, so that $\chi(f)^H \simeq \chi(f^H)$ is an equivalence, as well. $\square$

Definition 2.9.11. Let G be a compactly metrizable group and let V be a d-dimensional complex G-representation. We may factor the classifying map of V as

$$* \xrightarrow{V} \mathrm{B}_G(U(d)) \to \mathrm{B}\mathbf{U}_G \to \mathbf{BUP}_G\,.$$

Applying the Thom spectrum functor $\mathrm{Th}_G : (\mathcal{S}_G)_{/\,\mathbf{BOP}_G} \to \mathrm{Sp}_G$ yields maps of Thom spectra

$$\mathbb{S}^V \xrightarrow{\mathrm{Th}_G(V)} \Sigma^\infty \mathrm{B}_G(U(d))^{\gamma_d} \xrightarrow{t^U_d} \mathrm{B}\mathbf{U}^\gamma_G \xrightarrow{i} \mathbf{MUP}_G\,.$$

The composite of the first two G-maps represent the *universal Thom class* $t^U(V) \in \pi^G_V(\mathrm{B}\mathbf{U}^\gamma_G)$ for V, while $t^U_d \in [\mathrm{B}_G(U(d))^{\gamma_d}, \mathbf{MUP}_G]^G$ can be thought of as the universal Thom class for the universal d-dimensional vector bundle.

Notation 2.9.12 (RO(G)-graded unit)**.** Let G be a compactly metrizable group. Let E be a commutative homotopy ring G-spectrum, i.e. a commutative algebra object in the homotopy category of G-spectra. A class $t(V) \in \pi^G_V(E)$ is an RO(G)-graded unit, if and only if, multiplication by $t(V)$ defines an equivalence

$$t(V)\cdot(-) : \mathbb{S}^V \otimes E \xrightarrow{t(V)\otimes \mathrm{id}} E \otimes E \xrightarrow{\mathrm{mult}} E$$

of G-spectra.

Note that $t(V)$ is an RO(G)-graded unit if and only if there exists a class $\tau(V) \in \pi^G_{-V}(E)$ such that external product

$$t(V)\cdot \tau(V) : \mathbb{S} \xrightarrow{\simeq} \mathbb{S}^V \otimes \mathbb{S}^{-V} \xrightarrow{t(V)\otimes\tau(V)} E \otimes E \xrightarrow{\mathrm{mult}} E,$$

is equal to the unit of the homotopy ring G-spectrum E in $\pi^G_0(E)$. The RO(G)*-graded inverse* $\tau(V) \in \pi^G_{-V}(E)$ of $t(V)$ is unique.

Lemma 2.9.13. Let G be a compactly metrizable group. Precompostion by the morphism

$$i : \mathrm{B}\mathbf{U}^\gamma_G \to \mathbf{MUP}_G$$

in $\mathrm{CAlg}(\mathrm{Sp}_G)$ induces a fully-faithful functor i^* on slice categories. A morphism

$$f : \mathrm{B}\mathbf{U}^\gamma_G \to E \qquad \text{in } \mathrm{CAlg}(\mathrm{Sp}_G) \text{ factors through} \qquad i : \mathrm{B}\mathbf{U}^\gamma_G \to \mathbf{MUP}_G$$

if and only if for all finite dimensional unitary G-representations V the *Thom class*

$$f_*(t^U(V)) \in \pi^G_V(E)$$

is an RO(G)-graded unit.

Proof. Precomposition by the group completion $j : \mathrm{B}\mathbf{U}_G \to \mathbf{BUP}_G$ yields a fully faithful functor

$$j^* : \mathrm{CMon}\,(\mathcal{S}_G)_{\mathbf{BUP}_G\,/} \to \mathrm{CMon}\,(\mathcal{S}_G)_{\mathrm{B}\mathbf{U}_G\,/}$$

with essential image consisting of those maps $f : \mathrm{B}_G(U) \to X$ admitting an inverse in the monoid $[\mathrm{B}\mathbf{U}_G, X]^G$. Slicing over $\mathbf{BOP}_G$ yields a fully-faithful functor

$$j^* : \Big(\mathrm{CMon}\,(\mathcal{S}_G)_{/\,\mathbf{BOP}_G}\Big)_{\mathbf{BUP}_G\,/} \to \Big(\mathrm{CMon}\,(\mathcal{S}_G)_{/\,\mathbf{BOP}_G}\Big)_{\mathrm{B}\mathbf{U}_G\,/}$$

with essential image consisting of factorizations of $\mathrm{B}_G(U) \to \mathbf{BOP}_G$ through morphisms $f : \mathrm{B}_G(U) \to X$ for which f admits an inverse in the monoid $[\mathrm{B}\mathbf{U}_G, X]^G$. Because the G-equivariant Thom spectrum functor $\mathrm{Th}_G : \mathrm{CMon}\,(\mathcal{S}_G)_{/\,\mathbf{BOP}_G} \to \mathrm{CAlg}(\mathrm{Sp}_G)$ is a left adjoint, precomposition by $i := \mathrm{Th}_G(j) : \mathrm{B}\mathbf{U}^\gamma_G \to \mathbf{MUP}_G$ defines a fully-faithful functor

$$i^* : \mathrm{CAlg}(\mathrm{Sp}_G)_{\mathbf{MUP}_G\,/} \to \mathrm{CAlg}(\mathrm{Sp}_G)_{\mathrm{B}\mathbf{U}^\gamma_G\,/}$$

as well. We denote a right adjoint of Th_G by q, so that a morphism $f : \mathrm{B}\mathbf{U}^\gamma_G \to E$ lies in the essential image of i^* if and only if its adjoint $f^\flat : \mathrm{B}\mathbf{U}_G \to q(E)$ is invertible in the monoid

$[B\mathbf{U}_G, X]^G$. By Lemma 2.9.10, this condition on f is equivalent to the condition that for any unitary G-representation V, the composite G-map

$$* \xrightarrow{V} B\mathbf{U}_G \xrightarrow{f^\flat} q(E)$$

represents a unit in $[*, q(E)]^G$. This precisely requires the existence of a map $f' : * \to q(E)$ such that $f + f' = 0$ in $[*, q(E)]^G$. In particular, the composite $* \xrightarrow{f'} q(E) \to \mathbf{BOP}_G$ classifies $-V \in \mathrm{RO}(G)$. Unraveling the adjunction, we conclude that $f^\flat_*(V) \in \pi_0^G(q(E))$ represents a unit if and only if there is a morphism of G-spectra $f'_V : \mathbb{S}^{-V} \to E$ (adjoint to f') such that the composite

$$\mathbb{S} \simeq \mathbb{S}^V \otimes \mathbb{S}^{-V} \xrightarrow{f_*(t^U(V)) \otimes f'_V} E \otimes E \xrightarrow{\text{mult}} E$$

is homotopic to the unit of E, i.e. if and only if $f_*(t^U(V)) \in \pi_V^G(E)$ is an $\mathrm{RO}(G)$-graded unit. □

Definition 2.9.14. Let G be a compactly metrizable group. A *complete flag of G* is a $\mathbb{N}$-indexed diagram

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$$

of G-linear isometric embeddings of unitary G-representations such that for all $n \in \mathbb{N}_{\geq 1}$ the orthogonal complement

$$\alpha_n := V_n - V_{n-1}$$

of V_{n-1} in V_n is irreducible and such that every irreducible complex G-representation is G-linearly isomorphic to α_n for infinitely many $n \in \mathbb{N}_{\geq 1}$. Moreover, we demand that $V_0 = \{0\}$ is zero dimensional.

For any compactly metrizable group G a complete flag for G exists by the Peter-Weyl theorem.

Construction 2.9.15. Let $\{V_n\}_{n\geq 0}$ be a complete flag for a compactly metrizable group G. We denote by $B\mathbf{U}_G^\gamma[V_*^{-1}]$ the colimit of the $\mathbb{N}$-indexed diagram

$$B\mathbf{U}_G^\gamma \xrightarrow{(-)\cdot t^U(\alpha_1)} B\mathbf{U}_G^\gamma \otimes S^{-V_1} \xrightarrow{(-)\cdot t^U(\alpha_2)} B\mathbf{U}_G^\gamma \otimes S^{-V_2} \xrightarrow{(-)\cdot t^U(\alpha_3)} B\mathbf{U}_G^\gamma \otimes S^{-V_3} \xrightarrow{(-)\cdot t^U(\alpha_4)} \cdots$$

of left $B\mathbf{U}_G^\gamma$-modules in G-spectra. Consider the collection

$$S := \left\{ B\mathbf{U}_G^\gamma \otimes S^V \xrightarrow{(-)\cdot t^U(V)} B\mathbf{U}_G^\gamma : \ V \text{ a unitary representation} \right\}$$

of $B\mathbf{U}_G^\gamma$-module maps. By Remark 2.6.17, the filtered colimit preserving homotopy group functors $\pi_n^H(-) : \mathrm{Sp}_G \to \mathrm{Ab}$ for $H \in \mathrm{Lie}(G)$ and $n \in \mathbb{Z}$ are jointly conservative. So, the same argument as in the proof of [HH13, Lemma 2.3.] shows that $B\mathbf{U}_G^\gamma[V_*^{-1}]$ is S-local.

Because the tensor product of left $B\mathbf{U}_G^\gamma$-modules preserves colimits in both variables, it follows that the component map $B\mathbf{U}_G^\gamma \to B\mathbf{U}_G^\gamma[V_*^{-1}]$ exhibits $B\mathbf{U}_G^\gamma[V_*^{-1}]$ as idempotent algebra in left $B\mathbf{U}_G^\gamma$-modules. By [Lur17, Proposition 4.8.2.9.], there exists a unique lift of the map $B\mathbf{U}_G^\gamma \to B\mathbf{U}_G^\gamma[V_*^{-1}]$ to the category $\mathrm{CAlg}(\mathrm{Mod}(\mathrm{Sp}_G, B\mathbf{U}_G^\gamma))$ of commutative algebras in left $B\mathbf{U}_G^\gamma$-modules. Moreover, it follows from the theory of idempotent algebras, see [Lur17, Proposition 4.8.2.10.] that $B\mathbf{U}_G^\gamma[V_*^{-1}]$ is the initial S-local object of

$$\mathrm{CAlg}(\mathrm{Mod}(\mathrm{Sp}_G, B\mathbf{U}_G^\gamma)) \simeq \mathrm{CAlg}(\mathrm{Sp}_G)_{B\mathbf{U}_G^\gamma/-}.$$

By Lemma 2.9.13, the map $i : B\mathbf{U}_G^\gamma \to \mathbf{MUP}_G$ is another initial S-local object of $\mathrm{CAlg}(\mathrm{Sp}_G)_{B\mathbf{U}_G^\gamma/-}$.

We summarize the previous discussion in the following theorem:

Theorem 2.9.16 (Periodic Bordism is Telescoping)**.** *Let G be a compactly metrizable group and $\{V_n\}_{n\geq 0}$ a complete flag for G. The component inclusion $\mathrm{B}\mathbf{U}_G^\gamma \to \mathrm{B}\mathbf{U}_G^\gamma[V_*^{-1}]$ exhibits the telescoping colimit $\mathrm{B}\mathbf{U}_G^\gamma[V_*^{-1}]$ as an idempotent algebra in left $\mathrm{B}\mathbf{U}_G^\gamma$-modules in G-spectra. The unique map*

$$\mathrm{B}\mathbf{U}_G^\gamma[V_*^{-1}] \to \mathbf{MUP}_G \qquad \textit{in the slice category} \qquad \mathrm{CAlg}(\mathrm{Sp}_G)_{\mathrm{B}\mathbf{U}_G^\gamma/}$$

is an equivalence.

2.9.2. *Telescoping Complex Bordism Spectrum.* Now that we understood the periodic complex bordism spectrum $\mathbf{MUP}_G$ as a telescope on the suspension spectrum

$$\mathrm{B}\mathbf{U}_G^\gamma = \bigoplus_{d\in\mathbb{N}} \Sigma^\infty \mathrm{B}_G(U(d))^{\gamma_d} \in \mathrm{CAlg}(\mathrm{Sp}_G),$$

we want a similar telescoping description of its summand $\mathbf{MU}_G$.

Definition 2.9.17. Let V be a d-dimensional unitary representation of a compactly metrizable group G. The virtual vector bundle $\gamma_d - V$ is defined as the map

$$\gamma_d - V : \mathrm{B}_G(U(d)) \simeq \mathrm{B}_G(U(d)) \times * \xrightarrow{(\gamma_d, -V)} \mathbf{BUP}_G \times \mathbf{BUP}_G \xrightarrow{\mathrm{add}} \mathbf{BUP}_G\,.$$

Since $\dim(\gamma_d - V) = 0$, the map $\gamma_d - V$ factors through $\mathbf{BU}_G^0$. Applying the symmetric monoidal Thom spectrum functor $\mathrm{Th}_G(-)$ yields a map of G-spectra

$$\frac{t_d^U}{t^U(V)} := \mathrm{Th}_G(\gamma_d - V): \ \Sigma^\infty \mathrm{B}_G(U(d))^{\gamma_d} \otimes \mathbb{S}^{-V} \to \mathbf{MU}_G$$

such that the equation $\frac{t_d^U}{t^U(V)} \cdot t^U(V) = \mathrm{Th}_G(\gamma_d) =: t_d^U$ holds in $[\mathrm{B}_G(U(d))^{\gamma_d}, \mathbf{MUP}_G]^G$.

For W an k-dimensional G-representation and $m := d + k$ the dimension of $V' := V \oplus W$, the equation

$$\frac{t_d^U}{t^U(V)} \cdot \frac{t_k^U}{t^U(W)} = \frac{t_m^U}{t^U(V')}$$

holds in $[\mathrm{B}_G(U(d))^{\gamma_d} \otimes \mathrm{B}_G(U(d))^{\gamma_k} \otimes \mathbb{S}^{-V'}, \mathbf{MU}_G]^G$.

Corollary 2.9.18. *Let G be a compactly metrizable group and $\{V_n\}_{n\geq 0}$ a complete flag for G with $\dim(V_n) = d_n$. Then, the maps $\frac{t_{d_n}^U}{t^U(V_n)}$ exhibit the complex bordism spectrum $\mathbf{MU}_G$ as the colimit of the following diagram*

$$\mathbb{S} \to \Sigma^\infty \mathrm{B}_G(U(d_1))^{\gamma_{d_1}} \otimes \mathbb{S}^{-V_1} \longrightarrow \Sigma^\infty \mathrm{B}_G(U(d_2))^{\gamma_{d_2}} \otimes \mathbb{S}^{-V_2} \longrightarrow \cdots \tag{49}$$

of G-spectra, where for $\alpha_n := V_n - V_{n-1}$, the transition map

$$\Sigma^\infty \mathrm{B}_G(U(d_{n-1}))^{\gamma_{d_{n-1}}} \otimes \mathbb{S}^{-V_{n-1}} \to \Sigma^\infty \mathrm{B}_G(U(d_n))^{\gamma_{d_n}} \otimes \mathbb{S}^{-V_n}$$

is induced by multiplication with the Thom class $t^U(\alpha_n) \in \pi_{\alpha_n}^G(\mathrm{B}\mathbf{U}_G^\gamma)$. (The multiplication map $\mathrm{B}\mathbf{U}_G^\gamma \otimes \mathbb{S}^{\alpha_n} \to \mathrm{B}\mathbf{U}_G^\gamma$ is restricted to the appropriate summands and tensored with $\mathbb{S}^{-V_n}$).

Proof. The telescoping colimit $\mathrm{B}\mathbf{U}_G^{\gamma_d}[V_*^{-1}]$ from Construction 2.9.15 splits as an $n \in \mathbb{Z}$ indexed direct sum, with each summand collecting the terms $\Sigma^\infty \mathrm{B}_G(U(d)) \otimes \mathbb{S}^{-V}$ with $\dim(V) + d = n$. Indeed, the telescope has no maps between summands corresponding to different $n \in \mathbb{Z}$. This direct sum decomposition is compatible with the equivalences

$$\mathrm{B}\mathbf{U}_G^\gamma[V_*^{-1}] \xrightarrow[\simeq]{2.9.16} \mathbf{MUP}_G \xrightarrow[\simeq]{(46)} \bigoplus_{d\in\mathbb{Z}} \mathbf{MU}_G \otimes \mathbb{S}^{2d}.$$

The direct summand of $\mathrm{B}\mathbf{U}_G^{\gamma_d}[V_*^{-1}]$ corresponding to $0 \in \mathbb{Z}$ is precisely the telescope that we consider in this Corollary 2.9.18, and the identification of this summand of $\mathrm{B}\mathbf{U}_G^{\gamma_d}[V_*^{-1}]$ with $\mathbf{MU}_G$ is via the maps $\frac{t^U_{d_n}}{t^U(V_n)}$. ☐

Definition 2.9.19. The *Thom class* of the universal d-dimensional G-equivariant vector bundle is defined as

$$t^{\mathbf{MU}}(\gamma_d) := \frac{t^U_d}{t^U(\mathbb{C}^d)} \in [\mathrm{B}_G(U(d))^{\gamma_d}, \mathbf{MU}_G \otimes \mathbb{S}^{2d}]^G.$$

Pulling back $t^{\mathbf{MU}}(\gamma_d)$ along the Thomification of a classifying map $\xi : X \to \mathrm{B}_G(U(d))$ yield a *Thom class*

$$t^{\mathbf{MU}}(\xi) := \mathrm{Th}_G(\xi)^*(t^{\mathbf{MU}}) \in \left[X^\xi, \mathbf{MU}_G \otimes \mathbb{S}^{2d}\right]^G$$

in the $2d$-th $\mathbf{MU}_G$-cohomology of the Thom space X^ξ of the complex vector bundle ξ. In particular, for $X = *$ and $\xi = V$ a d-dimensional unitary G-representation, the Thom class

$$t^{\mathbf{MU}}(V) := \mathrm{Th}_G(V)^*(t^{\mathbf{MU}}(\gamma_d)) = \frac{t^U(V)}{t^U(\mathbb{C}^d)} \in \left[\mathbb{S}^V, \mathbf{MU}_G \otimes \mathbb{S}^{2d}\right]^G$$

is an $\mathrm{RO}(G)$-graded unit, with inverse $\frac{t^U(\mathbb{C}^d)}{t^U(V)} \in \pi^G_{d-V}(\mathbf{MU}_G)$. The equation

$$\frac{t^U_d}{t^U(V)} = t^{\mathbf{MU}}(\gamma_d) \cdot t^{\mathbf{MU}}(V)^{-1} \qquad \text{holds in} \qquad [\mathrm{B}_G(U(d))^{\gamma_d} \otimes S^{-V}, \mathbf{MU}_G]^G. \tag{50}$$

Remark 2.9.20. Suppose surjective morphisms $G \twoheadrightarrow G_n$ exhibit a topological group as an $\mathbb{N}^{\mathrm{op}}$-indexed limit of compact Lie groups G_n. Let $\{V_n\}_{n\geq 1}$ be a complete flag of G. As promised in Remark 2.8.3, we now provide an alternative argument that the homotopy ring G-spectra

$$\mathrm{Res}_G(\mathbf{MUP}) := \mathrm{colim}_n \, \mathrm{Infl}^G_{G_n}(\mathbf{MUP}_{G_n}) \qquad \text{and} \qquad \mathrm{B}\mathbf{U}^\gamma_G[V_*^{-1}]$$

agree, where we take the global ring spectrum $\mathbf{MUP}$ from [Sch18, Example 6.1.53.]. For each $n \in \mathbb{N}$, let $\{(V^n)_i\}_{i\geq 0}$ be a complete flag of G_n. The preferred map $\mathrm{B}\mathbf{U}^\gamma_{G_n}[(V^n)_*^{-1}] \to \mathbf{MUP}_{G_n}$ is an equivalence of homotopy ring G_n-spectra by the unitary analog of [Sch18, Theorem 6.1.23.]. We obtain maps

$$\mathrm{colim}_n \, \mathrm{Infl}^G_{G_n}(\mathbf{MUP}_{G_n}) \xrightarrow{\simeq} \mathrm{colim}_n \, \mathrm{Infl}^G_{G_n} \mathrm{B}\mathbf{U}^\gamma_{G_n}[(V^n)_*^{-1}] \to \mathrm{B}\mathbf{U}^\gamma_G[V_*^{-1}]$$

of homotopy ring G-spectra. The right map is an equivalence by cofinality, using the colimit formula for G-equivariant classifying spaces Equation (27) and the fact that any unitary G-representations is inflated from G_n for some $n \in \mathbb{N}$ by Theorem 2.1.4.

Note that this argument didn't need the existence of a symmetric monoidal Thom spectrum functor. At most, we needed the point-set level construction of Thom spaces that is necessary to make sense of the homotopy ring G-spectrum $\mathrm{B}\mathbf{U}^\gamma_G$. Moreover, the same argument as above works to prove an equivalence of homotopy ring spectra from the telescope in Equation (49) to the equivariant complex bordism spectrum

$$\mathrm{Res}_G(\mathbf{MU}) := \mathrm{colim}_n \, \mathrm{Infl}^G_{G_n}(\mathbf{MU}_{G_n}).$$

Again this argument is independent of Brink's upcoming results on equivariant Thom spectra.

3. Universality of Equivariant Complex Bordism

For the entire Section 3 we fix a compact abelian Hausdorff group A, whose Pontryagin dual $A^* := \hom_{\mathrm{Grp}}(A, U(1))$ is countable. Equivalently, as explained in Example 2.1.6, A is a topological group isomorphic to an inverse limit

$$\varprojlim_{n\in\mathbb{N}} \left(A_0 \xleftarrow{\phi_1} A_1 \xleftarrow{\phi_2} A_2 \xleftarrow{\phi_3} A_3 \xleftarrow{\phi_4} \dots\right) \tag{51}$$

for some diagram of compact abelian Lie groups $A_i \in \mathrm{CptLie}$. The category of these A is denoted $\mathrm{Ab(CptMet)}$, that is, $\mathrm{Ab(CptMet)}$ denotes the full subcategory of the category of topological groups spanned by compactly metrizable abelian groups.

Notation 3.0.1. For $E \in \mathrm{Sp}_A$ we write E_B for $\mathrm{res}_B(X)$ and for $X \in \mathcal{S}_{A,*}$ we abbreviate $\Sigma^\infty X$ by X, when the context is clear. We write $F(X, E)$ for the internal Hom-functor of Sp_A and $[X, E]^A$ for π_0 of the mapping spectrum $\mathrm{map}_{\mathrm{Sp}_A}(X, E)$.

Notation 3.0.2. We denote the unit of the symmetric monoidal category of genuine spectra by $\mathbb{S} := \Sigma^\infty_+(*) \in \mathrm{Sp}_A$.

Notation 3.0.3 (Notation (Co)-Homology)**.** For $E \in \mathrm{Sp}_A$ an A-spectrum and X either a pointed A-space or another A-spectrum, we denote by

- $E^A_n(X) := \pi^A_n(E \otimes X)$ the n-th E-homology of X.
- $E^n_A(X) := \pi^A_{-n} F(X, E) \cong [X, \mathbb{S}^n \otimes E]^A$ the n-th E-cohomology of X.
- For a map of A-spaces $X \to Y$ we write $E^*_A(Y, X) := E^*_A(Y/X)$ for the E-cohomology of the cofiber $\mathrm{cofib}(X \to Y) \in \mathcal{S}_{A,*}$.

3.1. Complex Orientability, Line Bundles and Projective Space.

Notation 3.1.1 (Equivariant Projective Space)**.** For a unitary A-representation V, we write $\mathbb{CP}(V) := S(V)/U(1)$ for the A-space of lines in V and γ_1 for the tautological A-equivariant complex line bundle on $\mathbb{CP}(V)$.

Note that when V is one-dimensional then $\mathbb{CP}(V)$ is a point.

Recollection 3.1.2. Recall from Example 2.5.24 that the tautological bundle γ_1 induces an equivalence between the space of lines $\mathbb{CP}(\mathcal{U}_A)$ in a complex A-universe $\mathcal{U}_A$ and the classifying space of A-equivariant line bundles $\mathbb{CP}^\infty_A := \mathrm{B}_A(U(1))$.

Notation 3.1.3. We write $\epsilon \in A^*$ for the trivial 1-dimensional unitary A-representation, so that $\mathbb{CP}(\epsilon) \to \mathbb{CP}^\infty_A$ represents the unit of the group object structure of $\mathrm{B}_A(U(1)) \in \mathrm{CMon}(\mathcal{S}_A)$.

A *commutative homotopy ring A-spectrum* is a commutative algebra object in the homotopy category of A-spectra, i.e. an A-spectrum $E \in \mathrm{Sp}_A$ together with a multiplication $E \otimes E \to E$ and a unit $\mathbb{S} \to E$ satisfying associativity, unitality and commutativity, up to homotopy.

Definition 3.1.4 (Complex Orientation [Col96])**.** Let E be a commutative homotopy ring A-spectrum. We call a class $x(\epsilon) \in E^2_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\epsilon))$ *complex orientation* of E if for all 1-dimensional unitary representation $\alpha \in A^*$, multiplication by the class

$$\mathrm{res}_{\epsilon\oplus\alpha}(x(\epsilon)) \in E^2_A(\mathbb{CP}(\epsilon \oplus \alpha), \mathbb{CP}(\epsilon))$$

defines an equivalence $E \otimes \mathbb{CP}(\epsilon \oplus \alpha)/\mathbb{CP}(\epsilon) \to E \otimes S^2$ of A-spectra. Moreover[11], we demand that, under the identification $\mathbb{CP}(\epsilon^2)/\mathbb{CP}(\epsilon) = \Sigma^2 S^0$ of pointed spaces, the class $\mathrm{res}_{\epsilon\oplus\epsilon}(x(\epsilon))$ is equal to the two-fold suspension of the ring unit of E.

To relate equivariant complex orientations to Thom spaces, we observe the following description of the Thom space of the homomorphism bundle $\hom(\gamma_1, V)$ as point-set quotient:

Lemma 3.1.5. The based A-equivariant homeomorphism

$$\mathbb{CP}(U)^{\hom(\gamma_1, V)} \to \mathbb{CP}(V \oplus U)/\mathbb{CP}(V), \qquad ([x_u], f) \mapsto [f(x_u) \oplus x_u] \tag{52}$$

is natural in the pair of unitary A-representations V and U with $\dim(V), \dim(U) < \infty$.

[11] [Col96] does not make this additional assumption.

By [Sch18, Proposition 1.1.19], the map $\mathbb{CP}(V) \to \mathbb{CP}(V \oplus U)$ is a Lie(A)-cofibration, so that Lemma 3.1.5 computes its cofiber in the ∞-category of A-spaces. Passing to the colimit along A-linear subspaces $U \subseteq \mathcal{U}_A$ of a complete complex A-universe, we conclude that the zero section s_0 fits into a cofiber sequence

$$\mathbb{CP}(V) \to \mathbb{CP}^\infty_A \xrightarrow{s_0} (\mathbb{CP}^\infty_A)^{\hom(\gamma_1, V)}$$

in the ∞-category of A-spaces.

Convention 3.1.6. For any pair of complex A-representations V and U with $\dim(V) < \infty$ we identify the cofiber $\mathbb{CP}(V \oplus U)/\mathbb{CP}(V)$ and $(\mathbb{CP}(U))^{\hom(\gamma_1, V)}$ as pointed A-spaces under $\mathbb{CP}(U)_+$.

Remark 3.1.7. The bundle $\hom(\gamma_1, \epsilon)$ is classified by an equivalence $(-)^{-1} : \mathbb{CP}^\infty_A \to \mathbb{CP}^\infty_A$, preserving the point $\mathbb{CP}(\epsilon)$. Moreover, the induced map on Thom spaces

$$(\mathbb{CP}^\infty_A)^{\hom(\gamma_1, \epsilon)} \to (\mathbb{CP}^\infty_A)^{\gamma_1}$$

is an equivalence of pointed A-spaces, as well. We obtain an equivalence $\mathbb{CP}^\infty_A/\mathbb{CP}(\epsilon) \simeq (\mathbb{CP}^\infty_A)^{\gamma_1}$ of pointed spaces under $(\mathbb{CP}^\infty_A)_+$.

Definition 3.1.8 (Thom Classes of Line Bundles)**.** Let $x(\epsilon) \in E^2_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\epsilon))$ be a complex orientation of a commutative homotopy ring A-spectrum E. We define the *Thom class* $t^E(\gamma_1) \in E^2_A({\mathbb{CP}^\infty_A}^{\gamma_1})$ of the universal A-equivariant line bundle γ_1 as the pullback of $x(\epsilon)$ along the above equivalence. More generally, for any A-equivariant line bundle $\xi : X \to \mathbb{CP}^\infty_A$ we define the *Thom class* $t^E(\xi) \in E^2_A(X^\xi)$ as the pullback of $t(\gamma_1)$ along the Thomification of the classifying map ξ.

By our convention, the equation $t^E(\hom(\gamma_1, \epsilon)) = x(\epsilon)$ holds in the group $E^2_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\epsilon))$. Hence, the relation $t^E(\hom(\gamma_1, \epsilon)) = \mathrm{Res}_{\epsilon\oplus\alpha}(x(\epsilon))$ holds in the group $E^2_A(\mathbb{CP}(\alpha)^{\hom(\gamma_1,\epsilon)})$. But, the bundle $\hom(\gamma_1, \epsilon)$ on the point $\mathbb{CP}(\alpha) = *$, identifies with the bundle $\alpha^{-1} := \hom(\alpha, \epsilon) \in A^*$ on the point $\mathbb{CP}(\epsilon) = *$. We conclude that the relation

$$t^E(\alpha^{-1}) = \mathrm{Res}_{\epsilon\oplus\alpha}(x(\epsilon)) \qquad \text{holds in } \; E^*_A(S^{\alpha^{-1}}). \tag{53}$$

We conclude:

Observation 3.1.9. A class $x(\epsilon) \in E^2_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\epsilon))$ in the cohomology of a commutative homotopy ring A-spectrum E is a complex orientation if and only if for all 1-dimensional complex A-representations α the Thom class

$$t^E(\alpha) \in \pi^A_{\alpha-2}(E) = E^2_A(S^\alpha) \cong E^2_A(\mathbb{CP}(\epsilon \oplus \alpha^{-1}), \mathbb{CP}(\epsilon))$$

is an RO(A)-graded unit, in the sense of Notation 2.9.12. Moreover, $t^E(\gamma_1)$ and $x(\epsilon)$ uniquely determine each other.

Example 3.1.10. The Thom class $t^{\mathbf{MU}}(\gamma_1) \in \mathbf{MU}^2_A({\mathbb{CP}^\infty_A}^{\gamma_1})$ from Definition 2.9.19 defines a complex orientation of the complex bordism spectrum $\mathbf{MU}_A \in \mathrm{CAlg}(\mathrm{Sp}_A)$.

The property that $t(\alpha)$ is an RO(A)-graded unit may be interpreted as a Thom isomorphism for 1-dimensional bundles over a point:

3.1.1. *Digression: The Thom Isomorphism.* For this digression we fix a complex orientation $x(\epsilon)$ of a commutative homotopy ring A-spectrum E. The *Thom homomorphism* is defined for an A-equivariant line bundle ξ over X as the following map

$$E \otimes X^\xi \xrightarrow{E\otimes\Delta} E \otimes X^\xi \wedge X_+ \xrightarrow{t(\xi)\wedge X} E \otimes S^2 \otimes X_+ \tag{54}$$

of A-spectra, where the *Thom diagonal* $\Delta : X^\xi \to X^\xi \wedge X_+$ is the Thomification of the bundle map $\xi \to \xi \times 0$.

Lemma 3.1.11. The Thom homomorphism is an equivalence of A-spectra.

Proof. To show that the Thom homomorphism is an equivalence in Sp_A for all equivariant line bundles $\xi : X \to \mathbb{CP}^\infty_A$ on X, we can reduce to the case of orbits $X = A/B$ with $B \in \mathrm{Lie}(A)$, by the passage to homotopy colimits. Thus, we may assume that the adjoint $\xi^\flat : * \to \mathrm{Res}^A_B(\mathbb{CP}^\infty_A)$ classifies a B-representation β. Employing the Restriction-Induction adjunction[12], we are reduced to showing that the adjoint of $t^E(\xi) : \mathrm{Ind}^A_B(S^\beta) \to E \otimes S^2$, denoted $t^E(\xi)^\flat : S^\beta \to E_B \otimes S^2$, is an $\mathrm{RO}(B)$-graded unit. Hence, it suffices to prove the following Lemma 3.1.12, as this means that $t^{E_B}(\xi^\flat) = t^E(\xi)^\flat$ is an $\mathrm{RO}(B)$-graded unit. □

Lemma 3.1.12. For any closed subgroup $B \leq A$, the class $\mathrm{Res}^A_B(x(\epsilon)) \in E^*_B(\mathbb{CP}^\infty_B, \mathbb{CP}(\epsilon))$ is a complex orientation of E_B.

Proof. The Pontryagin dual $B^* \to A^*$ of the inclusion $A \hookrightarrow B$ is surjective, so for any $\beta \in B^*$, we may choose $\alpha \in A^*$ with $\mathrm{Res}^A_B\, \alpha = \beta$. By assumption, the Thom class $t(\alpha)$ is an $\mathrm{RO}(A)$-graded unit, and therefore so is the Thom class $t(\beta) = \mathrm{Res}^A_B(t(\alpha))$. □

Remark 3.1.13. Our Thom isomorphism for equivariant line bundles and the identification of the pointed A-spaces $\mathbb{CP}(\alpha \oplus V)/\mathbb{CP}(\alpha)$ and $\mathbb{CP}(V)^{\hom(\gamma_1,\alpha)}$ under $\mathbb{CP}(V)_+$ allows us to compute $E^A_*(\mathbb{CP}(V)_+)$ by induction on $\dim(V)$. We instead present the original arguments, see [CGK00].

3.1.2. *Cohomology of Projective Space.* To explicitly describe the cohomology of projective space $E^*_A(\mathbb{CP}^\infty_A)$ we need two ingredients. First we produce even more classes.

Definition 3.1.14 (Change of base Point)**.** Let $x(\epsilon)$ be a complex orientation of $E \in \mathrm{Sp}_A$. For any $\alpha \in A^*$, we define $x(\alpha) \in E^*_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\alpha))$ as the pullback of $x(\epsilon)$ along the A-equivariant map

$$(-) \otimes \alpha^{-1} :\ \mathbb{CP}^\infty_A/\mathbb{CP}(\alpha) \to \mathbb{CP}^\infty_A/\mathbb{CP}(\epsilon)$$

induced by tensoring with α^{-1} on cofibers.

Remark 3.1.15. We have $x(\alpha) = t^E(\hom(\gamma_1, \alpha))$ in $E^*_A(\mathbb{CP}^\infty_A/\mathbb{CP}(\alpha)) = E^*_A(\mathbb{CP}^\infty_A)^{\hom(\gamma_1,\alpha)}$. Indeed, under the identification from Lemma 3.1.5, the map $(-)\otimes\alpha^{-1}$ corresponds to the induced map on Thom-spaces by the canonical bundle morphism $\hom(\gamma_1, \alpha) \to \hom(\gamma_1, \epsilon)$.

Construction 3.1.16 (Coordinates)**.** If a unitary A-representation $V = \alpha_1 \oplus \cdots \alpha_n$ splits as a direct sum of 1-dimensional representations $\alpha_i \in A^*$, then the bundle map

$$\hom(\gamma_1, V) \to \hom(\gamma_1, \alpha_1) \times \cdots \times \hom(\gamma_1, \alpha_1)$$

induces a map

$$(\mathbb{CP}^\infty_A)^{\hom(\gamma_1,V)} \to (\mathbb{CP}^\infty_A)^{\hom(\gamma_1,\alpha_1)} \wedge (\mathbb{CP}^\infty_A)^{\hom(\gamma_1,\alpha_2)} \wedge \cdots \wedge (\mathbb{CP}^\infty_A)^{\hom(\gamma_1,\alpha_n)}$$

on Thom spaces. Let $x(\epsilon)$ be a complex orientation of $E \in \mathrm{Sp}_A$. We denote the pullback of the exterior product $x(\alpha_1) \wedge \cdots \wedge x(\alpha_n)$ by

$$x(V) \in E^{2n}_A(\mathbb{CP}^\infty_A)^{\hom(\gamma_1,V)} = E^{2n}_A(\mathbb{CP}^\infty_A, \mathbb{CP}(V)).$$

We define the *coordinate* $y(V) \in E^{2n}_A\left(\mathbb{CP}^\infty_{A\,+}\right)$ as the class obtained from $x(V)$ by forgetting the subspace. Observe that in $E^*_A\left(\mathbb{CP}^\infty_{A\,+}\right)$

$$y(V) = y(\alpha_1) \cdots y(\alpha_n) \qquad \text{and} \qquad y(\alpha_i) = ((-) \otimes \alpha_i^{-1})^* y(\epsilon). \tag{55}$$

[12] For details, see the proof of Lemma 3.5.8.

Now, that we have enough classes, the second ingredient is a filtration of $\mathbb{CP}^\infty_A$. To this end, we choose a complete flag $\{V_n\}_{n\geq 0}$ for A, in the sense of Definition 2.9.14. For each $n \geq 1$, we denote the orthogonal complement of V_{n-1} in V_n by $\alpha_n \in A^*$. The advantage of having chosen a complete flag is the canonical equivalence $\operatorname{colim}_n \mathbb{CP}(V_n) \to \mathbb{CP}^\infty_A$ of A-spaces.

In Lemma 3.1.5, we constructed a cofiber sequence

$$\mathbb{CP}(V_n)_+ \to \mathbb{CP}(V_{n+1})_+ \to \mathbb{CP}(\alpha_{n+1})^{\hom(\gamma_1, V_n)}$$

of pointed A-spaces. We identify $\mathbb{CP}(\alpha_{n+1})^{\hom(\gamma_1,V_n)}$ with $S^{\alpha_{n+1}^{-1}\otimes V_n}$. If $(E, x(\epsilon))$ is a complex oriented A-spectrum and $n \geq 1$ an integer, we get a commutative[13] diagram

$$\begin{array}{ccccc} E\otimes \mathbb{CP}(V_n)_+ & \longrightarrow & E\otimes \mathbb{CP}(V_{n+1})_+ & \longrightarrow & E\otimes S^{\alpha_{n+1}^{-1}\otimes V_n} \\ \downarrow{\scriptstyle \bigoplus_{i=0}^{n-1} y(V_i)} & & \downarrow{\scriptstyle \bigoplus_{i=0}^{n} y(V_i)} & & \downarrow{\scriptstyle t(\alpha_{n+1}^{-1}\alpha_1)\wedge\cdots\wedge t(\alpha_{n+1}^{-1}\alpha_n)} \\ \bigoplus_{i=0}^{n-1} E[2i] & \longrightarrow & \bigoplus_{i=0}^{n} E[2i] & \longrightarrow & E[2n] \end{array}$$

where by convention $y(V_0) = y(0)$ is the unit of E. By assumption, see Observation 3.1.9, the right hand vertical map is an equivalence. We conclude by induction on $n \geq 1$, that the middle vertical map is an equivalence (of homotopy E-modules) in Sp_A.
Similarly, we apply $F(-, E)$ to the same cofiber sequence of pointed A-spaces and do an analogous induction, to see that

$$\bigoplus_{i=0}^{n} E[-2i] \xrightarrow{\bigoplus_{i=0}^{n} y(V_i)} F(\mathbb{CP}(V_{n+1})_+, E) \tag{56}$$

is an equivalence in Sp_A. We have proven the first statement of the following Proposition:

Proposition 3.1.17 (Cohomology of Projective Space, [Col96])**.** *Suppose $(E, x(\epsilon))$ is a complex oriented A-spectrum and $\{V_n\}_{n\geq 0}$ a complete flag of A. The pullback $E^*_A\left(\mathbb{CP}^\infty_{A}{}_+\right) \to E^*_A\left(\mathbb{CP}(V_{n+1})_+\right)$ sends the $(n+1)$-classes $y(0), y(V_1), \cdots, y(V_n)$ to an E^*_A-module basis. The ring map*

$$E^*_A\left(\mathbb{CP}^\infty_{A}{}_+\right) \to \varprojlim_n E^*_A\left(\mathbb{CP}(V_n)_+\right) \tag{57}$$

is an isomorphism. Any element $z \in E^*_A(\mathbb{CP}^\infty_{A}{}_+)$ can be expressed as

$$z = \sum_{i=0}^{\infty} a_i \cdot y(V_i) \qquad \text{for unique } a_i \in E^A_*. \tag{58}$$

The kernel of the projection $E^*_A\left(\mathbb{CP}^\infty_{A}{}_+\right) \to E^*_A\left(\mathbb{CP}(V_{n+1})_+\right)$ is generated as an ideal by $y(V_{n+1})$.

Proof. The second part of this proposition follows because the $\lim^1$-term of the relevant Milnor sequence vanishes. We conclude that the homomorphism $E^*_A(\mathbb{CP}^\infty) \to \prod_{i\in\mathbb{N}} E^*_A$ whose i-th component extracts the coefficient of the basis element $y(V_i)$ in $\mathbb{CP}(V_{i+k})$ ($k \geq 1$) is an E^*_A-module isomorphism. We obtain the unique expression (58). By its very construction, the class $y(V_{n+1})$ lies in the kernel of the projection $E^*_A\left(\mathbb{CP}^\infty_{A}{}_+\right) \to E^*_A\left(\mathbb{CP}(V_{n+1})_+\right)$. Conversely, by Equation (55), the class $y(V_{n+1})$ divides $y(V_i)$ for $i \geq n+1$, so $y(V_{n+1})$ divides any element in the kernel of $E^*_A\left(\mathbb{CP}^\infty_{A}{}_+\right) \to E^*_A\left(\mathbb{CP}(V_{n+1})_+\right)$. □

[13] By construction $x(\alpha_i)$ pulls back to $t^E(\hom(\gamma_1, \alpha_i)) = t^E(\alpha_{n+1}^{-1}\alpha_i)$ on $\mathbb{CP}(\alpha_{n+1}) = *$.

Convention 3.1.18. If E is a complex orientable commutative homotopy ring A-spectrum, we view $E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)$ as a complete topological ring, linearly topologized via the isomorphism in Equation (57).

The freeness of $E \otimes \mathbb{CP}(V_n)$ and $F(\mathbb{CP}(V_n), E)$, which we have seen before the proof of Proposition 3.1.17, imply that the universal coefficient map of E applied to $\mathbb{CP}^\infty_A$, as well as the following two Künneth maps, are isomorphisms.

Corollary 3.1.19 ((Co)homology of Products of Projective Spaces)**.** *Let E be a complex orientable A-spectrum. The homological Künneth map*

$$E_*^A\left(\mathbb{CP}^\infty_{A\,+}\right)^{\otimes n} \longrightarrow E_*^A\left((\mathbb{CP}^\infty_A)^n_+\right)$$

is an isomorphism of E_*^A-algebras. The cohomological Künneth map

$$E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)^{\otimes n} \to E_A^*\left((\mathbb{CP}^\infty_A)^n_+\right)$$

induces an isomorphism

$$E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)^{\hat{\otimes} n} \xrightarrow{\text{Künneth}} E_A^*\left((\mathbb{CP}^\infty_A)^n_+\right)$$

of completely linearly topologized E_A^*-algebras.

3.2. **Complex Orientation and Equivariant Formal Group Laws.**

3.2.1. *Motivation.* The definition of an A-equivariant formal group law is tailor-made to capture the structure and properties of $E_A^*(\mathbb{CP}^\infty_A)$, for E complex oriented. However, the notion should be entirely algebraic. In particular, an A-equivariant formal group law can be defined over any graded commutative ring k. Observe that the fixed points $(\mathbb{CP}^\infty_A)^A$ identify with

$$A^* \times \mathbb{CP}^\infty = \coprod_{\alpha \in A^*} \{\alpha\} \times \mathbb{CP}^\infty$$

as a group object in spaces $\mathcal{S}$. This can either be seen in the model, via the homeomorphism

$$A^* \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty(\mathcal{U}_A)^A, \qquad (\alpha, l) \mapsto \alpha \otimes l,$$

or via adjunction, by inspecting the mapping spaces $\mathrm{B_{gl}}A_n \to \mathrm{B_{gl}}U(1)$ in the global orbit category Glo. The Künneth isomorphism Corollary 3.1.19 implies that $E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)$ becomes a co-group object in linearly topologized graded E_A^*-algebras. Moreover, the map[14] of homotopy group objects $A^* \to \mathbb{CP}^\infty_A$ induces a morphism of co-group objects in linearly topologized graded E_A^*-algebras:

$$\theta : E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right) \to E_A^*(A^*) = \prod_{\alpha \in A^*} E_A^* = (E_A^*)^{A^*} \tag{59}$$

To see the continuity, note that under Definition 2.9.14, θ is the inverse limit of the maps $E_A^*\left(\coprod_{i=0}^n \mathbb{CP}(\alpha_i) \to \mathbb{CP}(V_i)\right)_+$ for a chosen complete flag $\{V_n\}_{n\geq 0}$ for A.

[14]Because $\mathbb{CP}^\infty$ is connected, this map is unique up to homotopy.

3.2.2. *Definition of Equivariant Formal Group Law.* For the remainder of this Section 3.2, we fix a graded commutative ring k. We think of k as $k = E_A^*$ for some complex oriented cohomology theory $E \in \mathrm{Sp}_A$. The topological ring $k^{A^*} = \prod_{A^*} k$ admits a continuous co-multiplication map

$$\Delta : k^{A^*} \to k^{A^* \times A^*} \cong k^{A^*} \hat{\otimes}_k k^{A^*}$$

which sends a function $f : A^* \to k$ to the function $\Delta(f) : A^* \times A^* \to k$ mapping (α, β) to $f(\alpha\beta) \in k$.[15] We endow the category of completely linearly topologized k-algebras with the structure of a symmetric monoidal category, via the completed tensor product $\hat{\otimes}$. The above discussion gives k^{A^*} the structure of a co-group object in that category.

Definition 3.2.1. For a k-algebra homomorphism $\theta : R \to k^{A^*}$ and $\alpha \in A^*$, we write $\theta(\alpha) : R \to k$ for the composite of θ with the projection to the α-factor. We call $\theta(\alpha)$ the *augmentation* at α. For a sequence $V_n = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in A^*$, we call the product ideal

$$I_{V_n} := \ker(\theta(\alpha_1)) \cdots \ker(\theta(\alpha_n)) \trianglelefteq R$$

the *augmentation ideal at* V_n. Note that

$$I_{V_n} = I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} \tag{60}$$

because the augmentation ideals are the preimage of coprime ideals.

We paraphrase [CGK00, Definition 11.1.]:

Definition 3.2.2 (Equivariant Formal Group Law)**.** A *(graded) A-equivariant formal group law* F is a tuple $(k, R, \theta, y(\epsilon))$ of a (graded) commutative ring k, a co-commutative co-group object in completely linearly topologized (graded) k-algebras R, a map of co-group objects $\theta : R \to k^{A^*}$ and a *coordinate* $y(\epsilon) \in R$ (homogenous of degree -2), such that the following hold:

(1) For some complete flag[16] of A the canonical homomorphism
$$R \to \varprojlim_{n\in\mathbb{N}} R/I_{V_n}$$
is an isomorphism of topological rings.

(2) The coordinate $y(\epsilon) \in R$ is regular, i.e. not a zero divisor, and generates the kernel of the augmentation $\theta(\epsilon)$.

Remark 3.2.3. We may replace condition (1) with either of the the following "choice free" conditions:

(1') the linear topology on R is generated by finite products of the augmentation ideals I_α for $\alpha \in A^*$.

(1") Condition (2) holds for every complete flag for A.

Remark 3.2.4. It is shown in [CGK00, Appendix B.] that the existence of a co-inverse can be removed from the Definition 3.2.2 of a A-equivariant formal group law. That is, it suffices to assume that R is an co-commutative co-monoid object. Even though their result is phrased for an abelian compact Lie group A, the assumption that the Pontryagin dual A^* is finitely generated is not needed: Their proof goes through for compactly metrizable abelian groups.

Example 3.2.5 (Formal Group Law associated to Complex Orientation)**.** Let

$$x(\epsilon) \in E_A^*(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$$

be a complex orientation of a commutative homotopy ring A-spectrum E. Let $y(\epsilon) \in E_A^*\left(\mathbb{CP}^\infty_{A+}\right)$ be the *coordinate* obtained from $x(\epsilon)$ by forgetting the base point. Set $k := \pi_*^A(E)$ and topologize

---

[15] The literature calls k^{A^*} the global sections of the constant formal group scheme associated to A^*.

[16] See Definition 2.9.14 for our definition of a complete flag.

$R := E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)$ as in Convention 3.1.18. By Proposition 3.1.17, the tuple $(k, R, \theta, y(\epsilon))$ is an A-equivariant formal group law, where the morphism θ was defined in Equation (59). When we invert the cohomological grading on $E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)$, then $(k, R, \theta, y(\epsilon))$ is a graded A-equivariant formal group law over the graded commutative ring $\pi_*^A(E)$.

3.2.3. *The underlying k-module of a Formal Group Law.* By passage to opposite categories, any A-equivariant formal group law defines a group object $\mathrm{Spf}(R)$ in formal k-schemes together with a group object morphism $\theta : A_k^* \to \mathrm{Spf}(R)$. Here A_k^* denotes the constant formal group scheme at the dual group A^*, base changed to k. In particular, A^* acts on $\mathrm{Spf}(R)$. Let us give this action a name:

Construction 3.2.6. Given a (graded) A-equivariant formal group law $F = (k, R, \theta, y(\epsilon))$, we construct an action of A^* on R: For any $\alpha \in A^*$

$$l_\alpha : R \xrightarrow{\Delta} R\hat{\otimes}_k R \xrightarrow{R\hat{\otimes}_k\theta(\alpha^{-1})} R,$$

defines a continuous k-algebra homomorphism such that $l_{\alpha\beta} = l_\alpha \circ l_\beta$ for all $\alpha, \beta \in A^*$ and $l_\epsilon = \mathrm{id}_R$. Moreover, for all $\alpha, \beta \in A^*$ we have $\theta(\alpha\beta^{-1}) = \theta(\alpha) \circ l_\beta$.

Topologically, the map $l_\alpha : E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right) \to E_A^*\left(\mathbb{CP}^\infty_{A\,+}\right)$ is induced by the A-map

$$(-) \otimes \alpha^{-1} : \mathbb{CP}^\infty_A \to \mathbb{CP}^\infty_A\,.$$

In Equation (55), we defined the coordinate $y(\alpha)$ by pulling back $y(\epsilon)$ along $(-) \otimes \alpha^{-1}$. We can mimick this construction algebraically.

Definition 3.2.7. Given a (graded) A-equivariant formal group law $F = (k, R, \theta, y(\epsilon))$. For $\alpha \in A^*$, we write $y(\alpha) \in R$ for the element $l_\alpha(y(\epsilon)) \in R$. Note that we have

$$y(\alpha\beta) = l_\alpha(y(\beta)) \qquad \text{for } \alpha, \beta \in A^* \tag{61}$$

The equation $\theta(\epsilon) = \theta(\alpha) \circ l_\alpha$ implies that $y(\alpha)$ is regular and generates the augmentation ideal $I_\alpha = \ker(\theta(\alpha))$. More generally, for a sequence $V_n = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in A^*$, we define the *coordinate* $y(V_n) := y(\alpha_1)\cdots y(\alpha_n)$. By Equation (60), $y(V_n) \in R$ is a regular element generating the augmentation ideal I_{V_n}.

Warning 3.2.8. In [Hau22], the author denotes l_V by $l_{V^{-1}}$, consequently $y(V)$ corresponds to $y(V^{-1})$.

For any elements $\alpha_1, \cdots, \alpha_n \in A^*$, the sequence of k-modules

$$0 \to R/y(\alpha_2)\cdots y(\alpha_n) \xrightarrow{\cdot y(\alpha_1)} R/y(\alpha_1)\cdots y(\alpha_n) \xrightarrow{\theta(\alpha_1)} k \to 0 \tag{62}$$

is split exact. We conclude by induction on $n \geq 1$, that the set of n elements

$$\{1,\ y(\alpha_1),\ y(\alpha_1)\cdot y(\alpha_2),\ \cdots, y(\alpha_1)\cdots y(\alpha_{n-1})\}$$

restrict to an k-module basis of $R/y(\alpha_1)\cdots y(\alpha_n) = R/I_{V_n}$. By condition (1) of an A-equivariant formal group law, for a chosen complete flag $\{V_n\}_{n\geq 0}$ for A, any element $z \in R$ can be expressed as

$$z = \sum_{i=0}^{\infty} a_i \cdot y(V_i) \qquad \text{for unique } a_i \in k, \tag{63}$$

just as in the "topological" setting (58).

3.2.4. *Euler Classes.* We will see that the Euler classes control the multiplicative structure of the underlying algebra of an A-equivariant formal group law. Moreover, the localization at Euler classes computes the geometric fixed points of a complex oriented cohomology theory.

Let's fix an (graded) equivariant formal group law $F = (k, R, \theta, y(\epsilon))$. While the k-module structure of R is determined, the A^*-action is not. For $\alpha \in A^* - \{\epsilon\}$, we might wonder how the points $\theta(\alpha) : \mathrm{Spec}(k) \to \mathrm{Spf}(R)$ and $\theta(\epsilon) : \mathrm{Spec}(k) \to \mathrm{Spf}(R)$ intersect. The scheme-theoretic intersection is computed as $\mathrm{Spec}(-)$ of the k-algebra $k \otimes_R k$, where k is an R-algebra via the augmentations $\theta(\alpha)$ on the left and $\theta(\epsilon)$ on the right. The k-algebra homomorphism

$$k \to k \otimes_R k, \qquad a \mapsto 1 \otimes_R a$$

is surjective with kernel generated by the *Euler class*

$$e_\alpha := \theta(\epsilon)(y(\alpha)) = \theta(\alpha^{-1})(y(\epsilon)). \tag{64}$$

We conclude that the scheme theoretic intersection is empty if and only if $e_\alpha \in k$ is a unit.

We denote the co-inverse of R by χ. The regular elements $\chi(y(\epsilon))$ and $y(\epsilon)$ differ by a unit in R, as they generate the same ideal I_ϵ. Because $e_\alpha = \theta(\alpha)(\chi(y(\epsilon))$, the Euler classes e_α and $e_{\alpha^{-1}}$ differ by a unit in k. We have $e_\alpha = 0$ if and only if $\theta(\alpha) = \theta(\epsilon)$:

Lemma 3.2.9. The Euler class $e_\alpha \in k$ vanishes if and only if there is a unit $u \in R^\times$ with $y(\alpha) = u \cdot y(\epsilon)$.

Proof. When $e_{\alpha^{-1}} = \theta(\alpha)(y(\epsilon)) = 0$, then $y(\alpha)$ divides $y(\epsilon) \in I_\alpha$. When, $e_\alpha = 0$, then $y(\epsilon)$ divides $y(\alpha)$. So, the regular elements $y(\alpha)$ and $y(\epsilon)$ mutually divide each other. □

If the formal group law F is associated to a complex orientation $x(\epsilon) \in E_A^*(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$, then e_α can be obtained from the Thom-class $t(\alpha) \in E_A^2(S^\alpha)$ by pulling back along the zero section $S^0 \to S^\alpha$.

3.3. Changing the Symmetry Group: Topologically and Algebraically.

3.3.1. *Topologically: Pushforward Orientation.* In Lemma 3.1.12 we discussed how to restrict complex orientations to subgroups. In this section, we will instead discuss the pushforward of a complex orientation of a commutative homotopy ring A-spectrum along $\varphi_* : \mathrm{Sp}_A \to \mathrm{Sp}_B$:

Convention 3.3.1. Throughout this Section 3.3, $\varphi : A \to B$ denotes a continuous group homomorphism. We denote the right-adjoint of the inflation-restriction functor $\varphi^* : \mathrm{Sp}_B \to \mathrm{Sp}_A$ by $\varphi_* : \mathrm{Sp}_A \to \mathrm{Sp}_B$.

For a complex oriented A-spectrum $(E, x(\epsilon))$, the pushforward of complex orientation, will allows us to pushforward the associated A-equivariant formal group law over $\pi_A^*(E)$ to an B-equivariant formal group law over $\pi_B^*(\varphi_* E)$. The identification $\pi_A^*(E) \cong \pi_B^*(\varphi_* E)$ will be our starting point to describe the pushforward (a.k.a. co-restriction) of equivariant formal group laws algebraically.

Construction 3.3.2 (Pushforward of Complex Orientations)**.** Let $x(\epsilon) \in E_A^2(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$ be a complex orientation of a commutative homotopy ring A-spectrum E. We define the *pushforward complex orientation* $\varphi_*(x(\epsilon)) \in (\varphi_* E)_B^2(\mathbb{CP}_B^\infty, \mathbb{CP}(\epsilon))$ as the adjoint of the composite A-map

$$\varphi^*(\mathbb{CP}_B^\infty)/\mathbb{CP}(\epsilon) \to \mathbb{CP}_A^\infty/\mathbb{CP}(\epsilon) \xrightarrow{x(\epsilon)} E \otimes S^2.$$

Lemma 3.3.3. In the situation of the previous Construction 3.3.2, the pushforward $\varphi_*(x(\epsilon))$ is a complex orientation of the commutative homotopy ring B-spectrum $\varphi_* E$.

Proof. In the following proof, we will denote the adjoint of any map $f : \varphi^* X \to Y$ by $f^\# : X \to \varphi_* Y$. Let $\alpha \in B^*$. By Observation 3.1.9, the composite

$$t(\alpha) : \varphi^* S^\alpha = \mathbb{CP}(\epsilon \oplus \varphi^* \alpha^{-1})/\mathbb{CP}(\epsilon) \to \mathbb{CP}^\infty_A/\mathbb{CP}(\epsilon) \xrightarrow{x(\epsilon)} E[2]$$

represents an $RO(A)$-graded unit. We choose an RO(A)-graded inverse $\tau(\alpha) \in E_2^A(\varphi^*(S^\alpha))$. To prove the lemma, we need to show that the composite

$$t(\alpha)^\# : S^\alpha = \mathbb{CP}(\alpha^{-1} \oplus \epsilon)/\mathbb{CP}(\epsilon) \to \mathbb{CP}^\infty_B/\mathbb{CP}(\epsilon) \xrightarrow{\varphi_* x(\epsilon)} \varphi_* E[2]$$

represents an $RO(B)$-graded unit. Note that $t(\alpha)^\#$ is the adjoint map of $t(\alpha)$, so our best guess for an inverse is to consider $\tau(\alpha)^\#$, where $\tau(\alpha)^\#$ is defined as the adjoint of a representative

$$\tau(\alpha) : \varphi^*(S^2) = S^2 \xrightarrow{\tau(\alpha)} E \otimes \varphi^*(S^2)$$

of the class $\tau(\alpha)$. We claim that the composite

$$S^2 \xrightarrow{\tau(\alpha)^\#} \varphi_*(E \otimes \varphi^*(S^\alpha)) \xrightarrow{\pi^{-1}} \varphi_* E \otimes S^\alpha$$

is the required RO(B)-graded inverse of $t(\alpha)^\#$, where π is the projection formula[17], which is an equivalence by [BDS16a, Proposition 2.15.]. We need to show that the left most composite from the top left corner to the bottom right corner in the commutative diagram

$$\begin{array}{ccccc}
(S^\alpha \otimes \mathbb{S}^{-2}) \otimes S^2 & = & S^\alpha \otimes \mathbb{S}^{-2} \otimes S^2 & \xrightarrow{((m\otimes 1)\circ(t(\alpha)\otimes\tau(\alpha))^\#} & \\
\downarrow{\scriptstyle t(\alpha)^\# \otimes \tau(\alpha)^\#} & & \downarrow{\scriptstyle (t(\alpha)\otimes\tau(\alpha))^\#} & & \\
\varphi_*(E) \otimes \varphi_*(E \otimes \varphi^* S^\alpha) & \xrightarrow{\text{lax}} & \varphi_*(E \otimes E \otimes \varphi^* S^\alpha) & \xrightarrow{\varphi_*(m\otimes 1)} & \varphi_*(E \otimes \varphi^* S^V) \\
\downarrow{\scriptstyle 1\otimes\pi^{-1}} & & \downarrow{\scriptstyle \pi^{-1}} & & \downarrow{\scriptstyle \pi^{-1}} \\
\varphi_*(E) \otimes \varphi_*(E) \otimes S^\alpha & \xrightarrow{\text{lax}\otimes 1} & \varphi_*(E \otimes E) \otimes S^\alpha & \xrightarrow{\varphi_*(m)\otimes 1} & \varphi_*(E) \otimes S^\alpha
\end{array}$$

is homotopic to $S^\alpha \otimes u^\#$. Here $u : \mathbb{S} \to E$ represents the unit of E and $m : E \otimes E \to E$ the multiplication. Because $\tau(\alpha)$ is defined as the RO(A)-graded inverse of the class $t(\alpha)$, the right-most composite is homotopic to $\pi^{-1} \circ (\varphi^* S^\alpha \otimes u)^\#$. It is left to be proven that $(\varphi^* S^\alpha \otimes u)^\#$ is homotopic to $\pi \circ (S^\alpha \otimes u^\#)$. In the following square the right-diagram commutes by naturality.

$$\begin{array}{ccccc}
\varphi_*(\varphi^*(S^\alpha \otimes \mathbb{S})) & \xrightarrow{\simeq} & \varphi_*(\varphi^*(S^\alpha) \otimes \varphi^*(\mathbb{S})) & \xrightarrow{\varphi_*(\varphi^* S^\alpha \otimes u)} & \varphi_*(\varphi^* S^\alpha \otimes E) \\
\uparrow & & \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} \\
S^\alpha \otimes \mathbb{S} & \longrightarrow & S^\alpha \otimes \varphi_* \varphi^*(\mathbb{S}) & \xrightarrow{S^\alpha \otimes \varphi_* u} & S^\alpha \otimes \varphi_* E
\end{array}$$

To check that the left square commutes pass to adjoint maps and use the triangle identity. Now, via first going vertically in the outer diagram we get $(\varphi^* S^\alpha \otimes u)^\#$. When we instead go horizontally first, we get $\pi \circ (S^\alpha \otimes u^\#)$. □

In the situation of Lemma 3.3.3, we have an A-equivariant graded formal group law $F = (k, R, \theta, y(\epsilon))$ with $k = \pi_A^* E_A$ and $R = E_A^{-*}\left(\mathbb{CP}^\infty_{A\,+}\right)$, associated to the orientation $x(\epsilon)$ of E. We also have an B-equivariant graded formal group law

$$\varphi_* F = (k,\, \varphi_* R,\, \varphi_* \theta,\, \varphi_*(y(\epsilon))),$$

associated to the pushforward complex orientation $\varphi_* x(\epsilon)$, where $\varphi_* R := (\varphi_* E)^{-*}\left(\mathbb{CP}^\infty_{B\,+}\right)$. In detail, $\varphi_*(y(\epsilon))$ is obtained from $\varphi_* x(\epsilon)$ by pulling back along $(\mathbb{CP}^\infty_B)_+ \to \mathbb{CP}^\infty_B/\mathbb{CP}(\epsilon)$ and $\varphi_* \theta$ denotes the pullback along $B^* \to \mathbb{CP}^\infty_B$.

[17]In the context of equivariant homotopy theory the projection formula is also known as the shearing isomorphism.

We want to describe how φ_*F arises from F, algebraically. To this end, we consider the k-algebra map

$$R = E_A^{-*}\left(\mathbb{CP}^\infty_{A\,+}\right) \to E_A^{-*}\left(\varphi^*\mathbb{CP}^\infty_{B\,+}\right) \cong (\varphi_*E)^{-*}\left(\mathbb{CP}^\infty_{B\,+}\right) = \varphi_*R. \tag{65}$$

induced by the A-map $\varphi^*\mathbb{CP}^\infty_B \to \mathbb{CP}^\infty_A$ and adjunction. We choose a complete flag $\{V_n\}_{n\geq 0}$ for B, as in Definition 2.9.14. From the explicit description Proposition 3.1.17 of the Cohomology of equivariant projective spaces, we conclude that

$$E_A^{-*}\left(\varphi^*\mathbb{CP}^\infty_{B\,+}\right) \to \varprojlim_n E_A^{-*}\left(\mathbb{CP}(\varphi^*V_n)_+\right) = \varprojlim_n R/y(\varphi^*V_n)$$

is a k-algebra isomorphism. In other words, φ_*R is isomorphic (under R) to the completion of R at the ideals

$$y(\alpha_1)\cdots y(\alpha_n) \qquad \text{for} \qquad \alpha_1, \cdots, \alpha_n \in \mathrm{Im}(B^* \xrightarrow{\varphi^*} A^*). \tag{66}$$

The continuous k-algebra homomorphism $\varphi_*\theta : \varphi_*R \to k^{B^*}$ is uniquely determined by making the diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad\theta\quad} & k^{A^*} \\ \big\downarrow & & \big\downarrow \\ \varphi_*R & \overset{\varphi_*\theta}{\dashrightarrow} & k^{B^*} \end{array} \tag{67}$$

commute. Similarly, the co-group structure of φ_*R is uniquely determined by the property that $R \to \varphi_*R$ is a morphism of co-group objects in completely linearly topologized k-algebras. Lastly, $R \to \varphi_*R$ sends $y(\epsilon)$ to $\varphi_*y(\epsilon)$.

3.3.2. *Co-Restriction of Equivariant Formal Group Laws.* One upshot of the previous discussion is that the we didn't needed to know whether the A-equivariant formal group law $F = (k, R, \theta, y(\epsilon))$ "came from homotopy theory": We could describe the co-restricted B-equivariant formal group law φ_*F anyways.

Construction 3.3.4. To analyze this functoriality we construct the category FGL, respectively $\mathrm{FGL}^{\mathrm{gr}}$, of equivariant (graded) formal group laws: We denote the subcategory of topological groups spanned by compactly metrizable abelian groups by Ab(CptMet). The objects of the category $\mathrm{FGL}^{(\mathrm{gr})}$ are pairs (A, F) of a group $A \in \mathrm{Ab(CptMet)}$ and an A-equivariant (graded) formal group law F. A morphism (A, F) to (B, F') consists of a continuous group homomorphism $\varphi : A \to B$ and a *morphism of (graded) equivariant formal group laws*

$$F = (k, R, \theta, y(\epsilon)) \to F' = (k', R', \theta', y(\epsilon)')$$

over φ: The morphism $F \to F'$ consists of a (graded) ring homomorphism $k \to k'$ and a morphism of co-group objects $R \to R'$ in completely linearly topologized (graded) k-algebras such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\ \theta\ } & k^{A^*} \\ \big\downarrow & & \big\downarrow \\ R' & \xrightarrow{\ \theta'\ } & (k')^{B^*} \end{array}$$

commutes and $y(\epsilon)$ is sent to $y(\epsilon)'$.

There is an obvious forgetful functor fgt : $\mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{Ab(CptMet)}$. We will see that this forgetful functor encodes co-restriction of equivariant formal group laws via its straigthening $\mathrm{CptAb} \to \mathrm{Cat}_{(2,1)}$, $A \mapsto A$-FGL.

Construction 3.3.5 (Co-Restriction)**.** For an A-equivariant (graded) formal group $F = (k, R, \theta, y(\epsilon))$ and a morphism $\varphi : A \to B$ in Ab(CptMet) we define a cocartesian lift $F \to \varphi_* F$ of φ mimicking the effect on equivariant formal group laws caused by pushing forward a complex orientation along φ. We define the *co-restricted* B-equivariant formal group law

$$\varphi_* F = (k, \varphi_* R, \varphi_* \theta, \varphi_* y(\epsilon))$$

together with a morphism $F \to \varphi_* F$ as follows: we let $R \to \varphi_* R$ be the completion of R at the ideals described in Equation (66). Then, $\varphi_* R$ admits a unique co-group structure in linearly topologized k-algebras such that $R \to \varphi_* R$ is a homomorphism of co-group objects. The co-group homomorphism $\varphi_* \theta$ is defined as the unique map making the diagram (67) commute. Lastly, to extend this to a morphism $F \to \varphi_* F$ in $\mathrm{FGL}^{(\mathrm{gr})}$ over φ we need to define $\varphi_* y(\epsilon)$ as the image of $y(\epsilon)$ under the completion $R \to \varphi_* R$.

The morphism $F \to \varphi_* F$ is cocartesian, because completion is left adjoint to the forgetful functor from completely linearly topologized k-algebras to linearly topologized k-algebras.

Definition 3.3.6 (Global Functoriality)**.** The straightening

$$\mathrm{Ab}(\mathrm{CptMet}) \to \mathrm{Cat}_{(2,1)}, \qquad A \mapsto A\text{-}\mathrm{FGL}^{(\mathrm{gr})}, \;\; (\varphi : A \to B) \mapsto (F \mapsto \varphi_* F),$$

of the co-cartesian forgetful functor $\mathrm{fgt} : \mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{Ab}(\mathrm{CptMet})$ sends $A \in \mathrm{Ab}(\mathrm{CptMet})$ to the category of (graded) A-equivariant formal group laws

$$A\text{-}\mathrm{FGL}^{(\mathrm{gr})} := \mathrm{FGL}^{(\mathrm{gr})} \times_{\mathrm{Ab}(\mathrm{CptMet})} \{A\}.$$

Remark 3.3.7. As we saw in Section 3.3.1, if an A-equivariant formal group law F is associated to a complex orientation $x(\epsilon)$ of $E \in \mathrm{Sp}_A$, then the B-equivariant formal group law, associated to the pushforward complex orientation $\varphi_* x(\epsilon)$ of $\varphi_* E \in \mathrm{Sp}_B$ is canonically isomorphic to $\varphi_* F$.

3.4. **Equivariant Lazard Rings and Inverse Limits of Groups.** For any compact abelian Lie group A, there exists a "universal A-equivariant formal group law", constructed by [CGK00, Cole, Kriz and Greenlees]. We will generalize their result to all groups of our setup (51).

3.4.1. *Base Change of Formal Group Laws: Algebraically.* To understand the base-change of equivariant formal group laws, we show that the forgetful functor $A\text{-}\mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{CRing}^{(\mathrm{gr})}$ is a cocartesian fibration.

Construction 3.4.1 (Base Change)**.** Consider a (graded) A-equivariant formal group law $F = (k, R, \theta, y(\epsilon))$ and a homomorphism $k \to k'$ of (graded) commutative rings. The tuple

$$k' \hat{\otimes}_k F := (k',\, k' \hat{\otimes}_k R,\, k' \hat{\otimes}_k \theta,\, 1 \hat{\otimes}_k y(\epsilon)),$$

defines a (graded) A-equivariant formal group law and $R \to k' \hat{\otimes}_k R$ extends $k \to k'$ to a *base-change* morphism $F \to k' \hat{\otimes}_k F$ in the category $A\text{-}\mathrm{FGL}^{(\mathrm{gr})}$. It is clear that any morphism $F \to F'$ of (graded) equivariant formal group laws over $k \to k'$ factors uniquely through the base-change $F \to k' \hat{\otimes}_k F'$. Because these base-change morphisms compose, they form cocartesian lifts of $k \to k'$ along the forgetful functor $\mathrm{fgt} : A\text{-}\mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{CRing}^{(\mathrm{gr})}$.

We claim that the straigthening of the forgetful functor $A\text{-}\mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{CRing}^{(\mathrm{gr})}$, denoted

$$\mathrm{CRing}^{(\mathrm{gr})} \to \mathrm{Cat}_{(2,1)}, \qquad k \mapsto A\text{-}\mathrm{FGL}_k, \;\; (k \to k') \mapsto (F \mapsto k' \hat{\otimes}_k F),$$

factors through $\mathrm{Set} \subset \mathrm{Cat}_{(2,1)}$. Indeed, by Equation (63), for $F, F' \in A\text{-}\mathrm{FGL}_k^{(\mathrm{gr})}$ there is at most one morphism $F \to F'$ in $A\text{-}\mathrm{FGL}_k^{(\mathrm{gr})}$, and any such morphism is an isomorphism.

Theorem 3.4.2 ([CGK00, 14.3.])**.** *For any compact abelian Lie group A, there is a (graded)-commutative ring L_A, called the Lazard Ring, co-representing the functor*

$$\mathrm{CRing}^{(\mathrm{gr})} \to \mathrm{Set}, \qquad k \mapsto A\text{-}\mathrm{FGL}_k$$

In particular, there is a universal A-equivariant formal group law $F^{\mathrm{uni}} \in A\text{-}\mathrm{FGL}_{L_A}$, so that the statement of Theorem 3.4.2 can be reformulated as follows: The functor

$$\mathrm{CRing}^{(\mathrm{gr})}_{L_A/} \to A\text{-}\mathrm{FGL}^{(\mathrm{gr})}, \qquad (L_A \to k) \mapsto k\hat{\otimes}_{L_A} F^{\mathrm{uni}}$$

is an equivalence of categories (over the forgetful functors to $\mathrm{CRing}^{(\mathrm{gr})}$).

In this thesis section, we extend Theorem 3.4.2 to all $A \in \mathrm{Ab}(\mathrm{CptMet})$. To do so, we need the following

Proposition 3.4.3. *Let $A = \varprojlim_{n\in N} A_n$ be an inverse limit in $\mathrm{Ab}(\mathrm{CptMet})$ with projections $\varphi_n : A \to A_n$, then*

$$A\text{-}\mathrm{FGL}^{(gr)} \to \varprojlim_n \left(A_n\text{-}\mathrm{FGL}^{(gr)}\right), \qquad F \mapsto ((\varphi_n)_* F)_n$$

is an equivalence of categories. In other words,

$$\mathrm{Ab}(\mathrm{CptMet}) \to \mathrm{Cat}_{(2,1)}, \qquad A \mapsto A\text{-}\mathrm{FGL}^{(gr)}, \tag{68}$$

preserves limits indexed by $\mathbb{N}^{\mathrm{op}}$.

We prove this Proposition 3.4.3 at the end of the section. For now, let us record the following consequence:

Construction 3.4.4. Suppose

$$A = \varprojlim_{n\in\mathbb{N}} \left(A_0 \xleftarrow{\phi_1} A_1 \xleftarrow{\phi_2} A_2 \xleftarrow{\phi_3} A_3 \xleftarrow{\phi_4} \dots\right) \tag{69}$$

is an inverse limit in $\mathrm{Ab}(\mathrm{CptMet})$ with each A_n a Lie group. Then, $(\phi_n)_* F^{\mathrm{uni}}$ is classified by a (graded) ring homomorphism $L_{A_{n-1}} \to L_{A_n}$. We can define the *Lazard ring* $L_A := \mathrm{colim}_n L_{A_n}$. The component maps $L_{A_n} \to L_A$ induce an equivalence of categories,

$$\mathrm{CRing}^{(\mathrm{gr})}_{L_A/} \to \varprojlim_{n\in\mathbb{N}^{\mathrm{op}}} \left(\mathrm{CRing}^{(\mathrm{gr})}_{L_{A_n}/}\right)$$

between the slice category and the inverse limit of slice categories.

The composite

$$A\text{-}\mathrm{FGL}^{(\mathrm{gr})} \xrightarrow{3.4.3} \varprojlim_n \left(A_n\text{-}\mathrm{FGL}^{(\mathrm{gr})}\right) \xrightarrow{3.4.2} \varprojlim_n \left(\mathrm{CRing}^{(\mathrm{gr})}_{L_{A_n}/}\right) \xrightarrow{\simeq} \mathrm{CRing}^{(\mathrm{gr})}_{L_A/} \tag{70}$$

is an equivalence over the forgetful functors to $\mathrm{CRing}^{(\mathrm{gr})}$. We have proven the following:

Theorem 3.4.5. *Let $A = \varprojlim_{n\in\mathbb{N}} A_n$ be an inverse limit of abelian compactly metrizable groups. The (graded) commutative ring $L_A := \mathrm{colim}_n L_{A_n}$ corepresents the functor*

$$\mathrm{CRing}^{(\mathrm{gr})} \to \mathrm{Set}, \qquad k \mapsto A\text{-}\mathrm{FGL}_k$$

that sends a (graded) commutative ring to the set of A-equivariant formal group laws over k.

To proof of Proposition 3.4.3 starts with the following

Lemma 3.4.6. Let k be a (graded) commutative ring and let $\lambda : \mathbb{N}^{\mathrm{op}} \to \mathrm{FGL}^{(\mathrm{gr})}$ be a diagram such that λ factors through $\mathrm{FGL}^{(\mathrm{gr})}_k$. Then, the limit of λ exists and is preserved by the forgetful functor $\mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{Ab}(\mathrm{CptMet})$.

Proof. Suppose $\lambda(n) = (A_n, F^n)$ with $F^n = (k, R_n, \theta_n, y(\epsilon)_n)$ an A_n-equivariant formal group law. Our strategy is to construct $\lim \lambda = (A, F)$ and the A-equivariant formal group law $F = (k, R, \theta, y(\epsilon))$ by hand: Here, we write $A := \varprojlim_n A_n \in \mathrm{Ab}(\mathrm{CptMet})$ and define $R := \varprojlim_n R_n$ to be the limit in the category of topological k-algebras. This inverse limit can be computed on underlying k-algebras. Moreover, R is completely linearly topologized: The topology is generated by the kernels of the composite

$$R \to R_n \to R_n/I_{\alpha_1} \cdots I_{\alpha_i} \tag{71}$$

where n runs through $\mathbb{N}$ and $\alpha_1, \ldots, \alpha_i$ run through A_n^*. We define θ as the composite

$$R = \varprojlim_{n\in\mathbb{N}} R_n \xrightarrow{\theta_n} \varprojlim_{n\in\mathbb{N}} k^{A_n^*} \xleftarrow{\cong} k^{A^*}.$$

Consider $\alpha \in A^*$. There is an $n \in \mathbb{N}$ so that some $\alpha(n) \in A_n^*$ lifts α along $A_n^* \to A^*$. For $m \geq n$, we denote by $\alpha(m) \in A_m^*$ the image of $\alpha(n)$ along $A_n^* \to A_m^*$. We define $y(\alpha) \in R$ as the unique element that is sent to $y(\alpha(m))$ under the projection $R \to R_m$ for all $m \geq n$. Because any left exact functor preserves split exact sequences, $y(\alpha) \in R$ is regular and generates the kernel of $\theta(\alpha)$. Now given $\alpha_1, \ldots, \alpha_i \in A^*$, we may choose $n \in \mathbb{N}$ and lifts $\alpha_1(n), \ldots, \alpha_i(n) \in A_n^*$ as before. Using the short exact sequence (62), we see by induction on $i \geq 1$ that $R \to R_n$ induces an isomorphism

$$R/y(\alpha_1) \cdots y(\alpha_i) \to R_n/y(\alpha_1) \ldots y(\alpha_i). \tag{72}$$

In particular, the element $y(\alpha_1) \cdots y(\alpha_i)$ generates the kernel of the composite in Equation (71). We conclude that $R \to \lim_{\alpha_1,\ldots,\alpha_i \in A^*} R/y(\alpha_1) \cdots y(\alpha_n)$ is an isomorphism.

We deduce that $R\hat{\otimes}_k R \to \varprojlim_n R_n \hat{\otimes}_k R_n$ is an isomorphism as well. We define the co-multipliction $\Delta : R \to R\hat{\otimes}_k R$ as the composite

$$R = \varprojlim_n R_n \xrightarrow{\Delta_n} \varprojlim_n R_n \hat{\otimes}_k R_n \cong R\hat{\otimes}_k R.$$

Together with a similar construction of a co-inverse, this makes $F = (k, R, \theta, y(\epsilon))$ into a (graded) A-equivariant formal group law. Moreover, $(A, F) \in \mathrm{FGL}^{(\mathrm{gr})}$ is indeed the limit of the functor $\lambda : \mathbb{N} \to \mathrm{FGL}^{(\mathrm{gr})}$ because any cone over λ provides unique morphisms into A and R, which are compatible with all structure. □

Proof of Proposition 3.4.3. We are given a functor $\mathbb{N}^{\mathrm{op}} \to \mathrm{Ab}(\mathrm{CptMet})$ sending n to A_n. We define the group $A := \varprojlim_n A_n$ and write $\varphi_n : A \to A_n$ for the projection. Our goal is to show that the comparison functor

$$c : A\text{-}\mathrm{FGL}^{(\mathrm{gr})} \to \varprojlim_n (A_n\text{-}\mathrm{FGL}^{(\mathrm{gr})}), \qquad F \mapsto ((\varphi_n)_* F)_n$$

is an equivalence, where the co-restriction $(\varphi_n)_* F$ was discussed in Construction 3.3.5. We proceed by constructing an inverse functor. The evaluations induce an equivalence of categories:

$$\mathrm{Fun}^{\mathrm{co}}(\mathbb{N}^{\mathrm{op}}, \mathrm{FGL}^{(\mathrm{gr})}) \times_{\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{CptAb})} \{n \mapsto A_n\} \to \varprojlim_n (A_n\text{-}\mathrm{FGL})$$

so that an object of $\varprojlim_n (A_n\text{-}\mathrm{FGL})$ is represented by a functor $\lambda : \mathbb{N}^{\mathrm{op}} \to \mathrm{FGL}^{(\mathrm{gr})}$, sending all morphism to cocartesian morphism, together with a natural isomorphism between the composite

$$\mathbb{N}^{\mathrm{op}} \xrightarrow{\lambda} \mathrm{FGL}^{(\mathrm{gr})} \xrightarrow{\mathrm{fgt}} \mathrm{CptAb}$$

and $n \mapsto A_n$. Because cocartesian morphisms in $\mathrm{FGL}^{(\mathrm{gr})}$ are sent to isomorphisms under $\mathrm{FGL}^{(\mathrm{gr})} \to \mathrm{CRing}^{(\mathrm{gr})}$, we can choose $k \in \mathrm{CRing}^{(\mathrm{gr})}$ so that λ is in the essential image of

$\mathrm{FGL}_k^{(\mathrm{gr})} \to \mathrm{FGL}^{(\mathrm{gr})}$. By Lemma 3.4.6, the limit $\lim \lambda \in \mathrm{FGL}^{(\mathrm{gr})}$ exists and canonically lies in $A\text{-}\mathrm{FGL}^{(\mathrm{gr})}$. This construction is natural in λ and assembles into a functor

$$r: \qquad \varprojlim_n \left(A_n\text{-}\mathrm{FGL}^{(\mathrm{gr})}\right) \to A\text{-}\mathrm{FGL}^{(\mathrm{gr})}, \qquad \lambda \mapsto \varprojlim_n \lambda(n).$$

For a fixed A-equivariant formal group law $F \in A\text{-}\mathrm{FGL}^{(\mathrm{gr})}$, the morphisms $F \to (\varphi_n)_* F$ from Construction 3.4.1 assemble into a morphism $F \to \lim_n((\varphi_n)_*F)$ in $\mathrm{FGL}^{(\mathrm{gr})}$. We may view this as a morphism $F \to r \circ c(F)$ in $A\text{-}\mathrm{FGL}^{(\mathrm{gr})}$ lifting the identity of $k \in \mathrm{CRing}^{(\mathrm{gr})}$. As $A\text{-}\mathrm{FGL}_k^{(\mathrm{gr})}$ is a groupoid, the morphism $F \to r \circ c(F)$ is an isomorphism. This construction is natural in $F \in A\text{-}\mathrm{FGL}^{(\mathrm{gr})}$ and assembles into a natural equivalence $\mathrm{id} \overset{\cong}{\Longrightarrow} r \circ c$.

Conversely, if we fix a point in $\varprojlim_n (A_n\text{-}\mathrm{FGL}^{(\mathrm{gr})})$ represented by $\lambda : \mathbb{N}^{\mathrm{op}} \to \mathrm{FGL}^{(\mathrm{gr})}$ as above, the projection $r(\lambda) := \lim_n \lambda(n) \to \lambda(n)$ in $\mathrm{FGL}^{(\mathrm{gr})}$ (which lies over φ_n in $\mathrm{Ab}(\mathrm{CptMet})$) induces a morphism $(\varphi_n)_* r(\lambda) \to \lambda(n)$ in $A_n\text{-}\mathrm{FGL}^{(\mathrm{gr})}$. Running through n, this assembles into a morphism $c(r(\lambda)) \to \lambda$ in $\varprojlim_n (A_n\text{-}\mathrm{FGL}^{(\mathrm{gr})})$. To see that $c(r(\lambda)) \to \lambda$ is an isomorphism, it suffices to check that each projection $(\varphi_n)_* r(\lambda) \to \lambda(n)$ is an isomorphism in $A_n\text{-}\mathrm{FGL}$. For each $n \in \mathbb{N}$, this morphism lifts the identity of $k \in \mathrm{CRing}^{(\mathrm{gr})}$ and $A_n\text{-}\mathrm{FGL}_k^{(\mathrm{gr})}$ is a groupoid. Thus, this morphism $c(r(\lambda)) \to \lambda$ is an isomorphism. We have thus produced a natural equivalence $c \circ r \overset{\cong}{\Longrightarrow} \mathrm{id}$. □

Lemma 3.4.7. Let $(E, x(\epsilon))$ be a complex oriented A-spectrum and let $h : E \to \tilde{E}$ be homomorphism of commutative homotopy ring A-spectra. Then $h(x(\epsilon)) \in \tilde{E}_A^*(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$ is a complex orientation of $\tilde{E}$.

The previous lemma follows immediately from the definition of complex orientation.

Remark 3.4.8 (Base Change of Formal Group Laws via Base Change of Complex Orientation)**.** In the situation of Lemma 3.4.7, let F denote the graded A-equivariant formal group law associated to the orientation $x(\epsilon)$ of E and let $\tilde{F}$ denote the graded A-equivariant formal group law associated to the orientation $h(x(\epsilon))$ of $\tilde{E}$. The homomorphism h induces a morphism $F \to \tilde{F}$ of A-equivariant formal group laws. Base Change yields a unique morphism

$$\pi_*^A \tilde{E} \hat{\otimes}_{\pi_*^A E} F \to \tilde{F} \qquad \text{in} \quad A\text{-}\mathrm{FGL}_k^{\mathrm{gr}}$$

under $F \in A\text{-}\mathrm{FGL}^{\mathrm{gr}}$. This morphism is an isomorphism in $A\text{-}\mathrm{FGL}^{(\mathrm{gr})}$ as $A\text{-}\mathrm{FGL}_k^{(\mathrm{gr})}$ is a groupoid.

3.5. **Cohomology of Grassmannians and Thom Classes.**

Convention 3.5.1. In this section Section 3.5, we generalize the computations of Cole, Kriz and Greenlees [CGK02] to a fixed group

$$A = \varprojlim_{n \in \mathbb{N}^{\mathrm{op}}} \left(A_0 \xleftarrow{\phi_1} A_1 \xleftarrow{\phi_2} A_2 \xleftarrow{\phi_3} A_3 \xleftarrow{\phi_4} \cdots\right)$$

where this inverse limit computed in topological groups and A_n is an abelian compact Lie group for each $n \geq 0$. The classifying spaces of A-equivariant d-dimensional complex vector bundles is denoted by $\mathrm{B}_A(U(d)) \in \mathcal{S}_A$.

Proposition 3.5.2 (Homology of Grassmannians, [CGK02, 2.2.])**.** *Let E be a complex orientable A-spectrum and let $d \geq 1$ be an integer. The composite of the Künneth map with the map induced on E-homology by the map classifying the external sum of d-many A-equivariant line bundles*

$$E_*^A \left(\mathbb{CP}^\infty_{A\,+}\right)^{\otimes d} \xrightarrow{\text{Künneth}} E_*^A \left((\mathbb{CP}_A^\infty)_+^{\times d}\right) \xrightarrow{\oplus_*} E_*^A(\mathrm{B}_A(U(d))_+) \tag{73}$$

induces an isomorphism

$$\{E_*^A\left(\mathbb{CP}_{A+}^\infty\right)^{\otimes d}\}_{\Sigma_d} \to E_*^A(\mathrm{B}_A(U(d))_+) \tag{74}$$

of E_*^A-modules from the Σ_d-coinvariants to the unreduced E-homology of $\mathrm{B}_A(U(d))$. In particular, $E_*^A(\mathrm{B}\mathbf{U}_{A+})$ is isomorphic to $\mathrm{Sym}\left(E_*^A\left(\mathbb{CP}_{A+}^\infty\right)\right)$ as a monoid in $\pi_*^A E$-modules.

Proof. We choose a complex orientation $x(\epsilon) \in E_A^2(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$ and write $\varphi_n : A \to A_n$ for the projection. Let us endow the A_n-spectrum

$$E_n := (\varphi_n)_* E \in \mathrm{Sp}_{A_n}$$

with the pushforward complex orientation $(\varphi_n)_*(x(\epsilon))$ from Construction 3.3.2. The projection formula, which holds by [BDS16b, Proposition 2.15.], defines an equivalence

$$E_n \otimes \mathrm{B}_{A_n}(U(d))_+ \xrightarrow{\pi} (\varphi_n)_* \left(E \otimes (\varphi_n)^* \mathrm{B}_{A_n}(U(d))_+\right)$$

and applying $\pi_*^{A_n}(-)$ on both sides yields an isomorphism

$$(E_n)_*^{A_n}(\mathrm{B}_{A_n}(U(d))_+) \cong E_*^A((\varphi_n)^* \mathrm{B}_{A_n}(U(d))_+).$$

From the equivalence

$$\mathrm{colim}_n (\varphi_n)^* \mathrm{B}_{A_n}(U(d)) \to \mathrm{B}_A(U(d))$$

of Equation (27), we get an isomorphisms

$$\mathrm{colim}_n (E_n)_*^{A_n}(\mathrm{B}_{A_n}(U(d))_+) \cong E_*^A(\mathrm{B}_A(U(d))_+).$$

An analogous construction provides an isomorphism $\mathrm{colim}_n (E_n)_*^{A_n}\left((\mathbb{CP}_{A_n}^\infty)_+^{\times d}\right) \cong E_*^A\left(\mathbb{CP}_{A\,+}^{\infty\,\times d}\right)$. Moreover, taking the colimit of the homomorphisms

$$(E_n)_*^{A_n}\left(\mathbb{CP}_{A_n+}^\infty\right)^{\otimes d} \xrightarrow{\text{Künneth}} E_*^{A_n}\left((\mathbb{CP}_{A_n}^\infty)_+^{\times d}\right) \xrightarrow{\oplus_*} E_*^{A_n}(\mathrm{B}_{A_n}(U(d))_+)$$

gives Equation (73). So, it suffices to see that the homomorphism from (74) is an isomorphism for each A_n, separately. This case is covered by [CGK02, Theorem 2.2.]. □

When we have chosen a complete flag of A, as in Definition 2.9.14, then we can be more explicit:

Lemma 3.5.3. Let $\beta(m) : E[2m] \to E \otimes \Sigma_+^\infty \mathbb{CP}_A^\infty$ denote the composite

$$E[2m] \xrightarrow{\text{incl}} \bigoplus_{i=0}^{m} E[2i] \xrightarrow{\simeq} E \otimes \mathbb{CP}(V_{m+1})_+ \to E \otimes \Sigma_+^\infty \mathbb{CP}_A^\infty\ ,$$

where we choose an inverse of the equivalence of homotopy E-modules constructed before Proposition 3.1.17. Then, the composite homotopy E-module map

$$\bigoplus_{0 \le i_1 \le i_2 \le \cdots \le i_d} E[*] \xrightarrow{\beta(i_1)\wedge\cdots\wedge\beta(i_d)} E \otimes \Sigma_+^\infty \left(\mathbb{CP}_A^\infty\right)^{\times d} \xrightarrow{\oplus_*} E \otimes \Sigma_+^\infty \mathrm{B}_A(U(d)) \tag{75}$$

is an equivalence in Sp_A. Here, $* = 2(i_1 + \cdots + i_d)$.

Proof. We write $\oplus_*(\beta(i_1) \wedge \cdots \wedge \beta(i_d)) \in E_*^A(\mathrm{B}_A(U(d))_+)$ for the image of $\beta(i_1) \otimes \cdots \otimes \beta(i_d)$ under the homomorphism (73). Because the homomorphism induced on co-invariants (74) is an isomorphism, the map (75) is an isomorphism after applying $\pi_A^*(-)$. For any $B \le A$ with $A/B \in \mathrm{Orb}_A$, the restriction $\mathrm{Res}_B^A(x(\epsilon))$ is a complex orientation of $E_B \in \mathrm{Sp}_B$, see Lemma 3.1.12. Doing the previous construction for the restricted orientation and restricted complete flag yields the classes $\mathrm{Res}_B^A(\beta(i_1) \wedge \cdots \wedge \beta(i_d))$. We conclude that the map (75) is an isomorphism after applying $\pi_B^*(-)$, too. The functors $\pi_B^*(-)$ are jointly conservative. Thus, the map of A-spectra (75) is an equivalence, as claimed. □

Corollary 3.5.4 (Cohomology of Grassmannians). *The composite of the map classifying the external sum of A-line bundles with the inverse of the Künneth map*

$$E_A^*(\mathrm{B}_A(U(d))_+) \xrightarrow{\oplus^*} E_A^*\left((\mathbb{CP}_A^\infty)_+^{\times d}\right) \xrightarrow{(K\ddot{u}nneth)^{-1}} E_A^*\left(\mathbb{CP}_{A\,+}^\infty\right)^{\hat{\otimes} d}$$

is an isomorphism onto its image $\{E_A^*\left(\mathbb{CP}_{A\,+}^\infty\right)^{\hat{\otimes} d}\}^{\Sigma_d}$, consisting of fixed points under the factor-permutation action.

Proof. The equivalence in Equation (75) implies that the universal coefficient map

$$E_A^*(\mathrm{B}_A(U(d))_+) \to \mathrm{Hom}_{E_A^*}(E_*^A(\mathrm{B}_A(U(d))_+), E_*^A)$$

is an isomorphism of E_A^*-algebras. Similarly, $E \otimes (\mathbb{CP}_A^\infty)_+^n$ is a free homotopy E-module, so the universal coefficient map is an isomorphism for $(\mathbb{CP}_A^\infty)^n$, too. Therefore, the result follows from the homological case stated in Proposition 3.5.2. □

To proceed with constructing Thom classes for complex vector bundles of dimension ≥ 2, we study the following map: For $\alpha_1 \in A^*$, viewed as a unitary A-representation, we have a map

$$(\alpha_1 \oplus -) : \ \mathrm{B}_A(U(d)) \to \mathrm{B}_A(U(d+1))$$

classifying the bundle $\alpha_1 \oplus \gamma_d$. Consider the augmentation, $\theta(\alpha_1) : * \to \mathbb{CP}_A^\infty$, classifying α_1, so that the diagram

$$\begin{array}{ccc}
\{E_*^A\left(\mathbb{CP}_{A\,+}^\infty\right)^{\hat{\otimes} d}\}_{\Sigma_d} & \xrightarrow{\{\theta(\alpha_1)\hat{\otimes}(-)\}_{\Sigma_{d+1}}} & \{E_*^A\left(\mathbb{CP}_{A\,+}^\infty\right)^{\hat{\otimes}(d+1)}\}_{\Sigma_{d+1}} \\
\cong\downarrow & & \downarrow\cong \\
E_*^A(\mathrm{B}_A(U(d))_+) & \xrightarrow{(\alpha_1\oplus(-))_*} & E_*^A(\mathrm{B}_A(U(d+1))_+)
\end{array}$$

commutes. If we extend $\alpha_1 \in A^*$ to a complete flag of A, then, in notation of Lemma 3.5.3, we have $\theta(\alpha_1) = \beta(0)$ in $E_A^*\left(\mathbb{CP}_{A\,+}^\infty\right)$. Therefore, the lower horizontal map of the above diagram, $(\alpha_1 \oplus (-))_*$, can be described on the basis of Lemma 3.5.3 by

$$\oplus_*\left(\beta(i_1) \wedge \cdots \wedge \beta(i_d)\right) \mapsto \oplus_*\left(\beta(0) \wedge \beta(i_1) \wedge \cdots \wedge \beta(i_d)\right).$$

In particular, the homomorphism $(\alpha_1 \oplus (-))_*$ is split injective.

Corollary 3.5.5. *Let E be a complex orientable A-spectrum. For any 1-dimensional A-representation $\alpha_1 \in A^*$, the cofiber sequence*

$$\mathrm{B}_A(U(d-1))_+ \xrightarrow{\alpha_1\oplus(-)} \mathrm{B}_A(U(d))_+ \xrightarrow{s_0} \mathrm{B}_A(U(d))^{\mathrm{hom}(\gamma_d,\alpha_1)} \tag{76}$$

of based A-spaces induces a split short-exact sequence

$$0 \to E_A^*(\mathrm{B}_A(U(d))^{\mathrm{hom}(\gamma_d,\alpha_1)}) \xrightarrow{s_0^*} E_A^*(\mathrm{B}_A(U(d))_+) \xrightarrow{(\alpha_1\oplus(-))_*} E_A^*(\mathrm{B}_A(U(d-1))_+) \to 0 \tag{77}$$

on E-cohomology.

Proof. The map $E_A^*(\mathrm{B}_A(U(d))_+) \xrightarrow{(\alpha_1\oplus(-))_*} E_A^*(\mathrm{B}_A(U(d-1))_+)$ is dual to a split injection, by the previous discussion. To obtain the cofiber sequence (76) we present three alternative arguments: One can compute the quotient in the model, see [CGK02, Remark 4.4.]. One can deduce the result from the compact Lie group case [CGK02, Remark 4.4.] via the passage to the colimit along inflations, or one used global spaces as follows: Applying the left adjoint functor $(-)_{/\!/U(d)} : \mathcal{S}_{U(d),*} \to \mathcal{S}_{\mathrm{gl},*}$ to the cofiber sequence $U(d)/U(d-1)_+ \to S^0 \to S^{\mathbb{C}^d}$ yields a cofiber sequence of pointed global spaces

$$\mathrm{B}_{\mathrm{gl}}U(d-1)_+ \to \mathrm{B}_{\mathrm{gl}}U(d)_+ \to S^{\mathbb{C}^d}_{/\!/U(d)}.$$

In turn, applying the restriction functor $\mathrm{Res}_A : \mathcal{S}_{\mathrm{gl},*} \to \mathcal{S}_{A,*}$ yields the cofiber sequence

$$\mathrm{B}_A(U(d-1))_+ \xrightarrow{(-)\oplus\epsilon} \mathrm{B}_A(U(d))_+ \xrightarrow{s_0} \mathrm{B}_A(U(d))^{\gamma_d} \simeq S^{\mathbb{C}^d} \wedge_{U(d)} \mathrm{B}_A(U(d)).$$

The maps classifying tensoring with α_1 provide an equivalence of this cofiber sequence of based A-spaces with the diagram in Equation (76). □

Construction 3.5.6. Let E be a complex oriented A-spectrum. The class $y(\epsilon)^{\times d} \in E_A^{2d}\left((\mathbb{CP}_A^\infty)_+^{\times d}\right)$ lifts to a unique class

$$t^E(\gamma_d) \in E_A^{2d}(\mathrm{B}_A(U(d))^{\gamma_d})$$

along the injection

$$E_A^*(\mathrm{B}_A(U(d))^{\gamma_d}) \xrightarrow{s_0^*} E_A^*(\mathrm{B}_A(U(d))_+) \xrightarrow{\oplus^*} E_A^*\left((\mathbb{CP}_A^\infty)_+^{\times d}\right).$$

Indeed, this follows from the short exact sequence in Equation (77): Note that the diagram

$$\begin{array}{ccc} E_*^A(\mathrm{B}_A(U(d))_+) & \xrightarrow{(\epsilon\oplus(-))^*} & E_*^A(\mathrm{B}_A(U(d-1))_+) \\ \cong\downarrow & & \downarrow\cong \\ \{E_A^*\left(\mathbb{CP}_{A}^\infty{}_+\right)^{\hat{\otimes}d}\}^{\Sigma_d} & \xrightarrow{\theta(\epsilon)\hat{\otimes}_{E_A^*}(-)} & \{E_A^*\left(\mathbb{CP}_{A}^\infty{}_+\right)^{\hat{\otimes}(d-1)}\}^{\Sigma_{d-1}} \end{array}$$

commutes and the class $y(\epsilon)$ lies in the kernel of $\theta(\epsilon) : E_A^*\left(\mathbb{CP}_{A}^\infty{}_+\right) \to E_A^*$.

Definition 3.5.7. Let E be a complex oriented A-spectrum. For a d-dimensional A-equivariant complex vector bundle classified by a map $\xi : X \to \mathrm{B}_A(U(d))$ we define the *Thom-class* $t^E(\xi) \in E_A^{2d}(X^\xi)$ as the pullback of $t^E(\gamma_d)$ along the Thomification of the classifying map $X^\xi \to \mathrm{B}_A(U(d))^{\gamma_d}$.

Note that for $V = \alpha_1 \oplus \cdots \oplus \alpha_d$ a bundle over a point with $\alpha_i \in A^*$, the class

$$t^E(V) = t^E(\alpha_1) \wedge \cdots \wedge t^E(\alpha_n) \in E_A^{2d}(S^V) \tag{78}$$

is an RO(A)-graded unit by Equation (53).

Lemma 3.5.8 (Thom Isomorphism)**.** *If E is a complex oriented A-spectrum and ξ an d-dimensional complex A-equivariant vector bundle on a cofibrant A-space X, then the Thom homomorphism*

$$E \otimes X^\xi \xrightarrow{E\otimes\Delta} E \otimes X^\xi \wedge X_+ \xrightarrow{t^E(\xi)\wedge X} E \otimes S^{2d} \otimes X_+ \tag{79}$$

is an equivalence in Sp_A. Here, the Thom diagonal Δ is the Thomification of the bundle map $\xi \to \xi \times 0$. This induces the Thom isomorphism

$$E_A^{*-2d}(X_+) \xrightarrow{t^E(\xi)\wedge(-)} E_A^*(X^\xi \wedge X_+) \xrightarrow{\Delta^*} E_A^*(X^\xi)$$

in E-cohomology.

Proof. The Thom isomorphism in E-cohomology follows from the spectral Thom isomorphism in Equation (79) because the spectral Thom isomorphism is homotopy E-linear and

$$E_A^n(-) = [-, E \otimes \mathbb{S}^n]^A \cong [E \otimes (-), E \otimes \mathbb{S}^n]^A_{E-\mathrm{linear}}.$$

Via passage to homotopy colimits, we may assume that X is an orbit A/B of a subgroup $B \leq A$ with $B \in \mathrm{Lie}(A)$. In this case, the classifying map $\xi : X \to \mathrm{B}_A(U(d))$ is adjoint to a classifying map $V : * \to \mathrm{B}_B(U(d))$ of a unitary B-representation V, under the restriction-induction adjunction. By Lemma 3.1.12, the orientation of E restricts to an orientation of $E_B := \mathrm{Res}_B^A(E)$. By the previous discussion, the Thom class $t^{E_B}(V) \in E_B^{2d}(S^V)$ of V is an

RO(B)-graded unit. By the left projection formula, a.k.a. the shearing isomorphism, the Thom homomorphism for ξ identifies with

$$\mathrm{Ind}_A^B\left(E_B \otimes S^V \xrightarrow[\simeq]{E\wedge t^{E_B}(V)} E_B \otimes S^{2d}\right),$$

using that $t^{E_B}(V) : S^V \to E_B$ is adjoint to $t^E(\xi) : \mathrm{Ind}_B^A(S^V) \to E$. □

Corollary 3.5.9. *Let E be a complex oriented A-spectrum. Then, the map $E^A \to \Phi^A(E)$ of homotopy ring spectra is a localization at the set of Euler classes $\{e_\alpha \in \pi_{-2}(E^A) : \alpha \in A^* \setminus \{\epsilon\}\}$.*

Proof. By Corollary 2.6.27, the map $E^A \to \Phi^A(E)$ is equivalent to the component inclusion

$$E^A \to \mathrm{colim}_{V \subseteq \mathcal{U}_A^\perp}(S^V \otimes E)^A,$$

where $\mathcal{U}_A$ denotes a complete complex A-universe, see Remark 2.6.26. For any complex A-representation V, the Thom isomorphism defines an equivalence

$$t^E(V)\cdot(-) : S^V \otimes E \xrightarrow{\simeq} S^{\dim_{\mathbb{R}}(V)} \otimes E.$$

If $V = W \oplus \alpha$ for $\alpha \subseteq \mathcal{U}_A^\perp$ one-dimensional, then the stabilization map $S^W \otimes E \to S^V \otimes E$ corresponds to multiplication by the Euler class

$$e_\alpha \cdot (-) : S^{\dim(W)} \otimes E \to S^2 \otimes S^{\dim_{\mathbb{R}}(W)} \otimes E$$

under the Thom isomorphism. Hence, $E^A \to \Phi^A(E)$ is homotopic to the telescoping localization of E^A at the Euler classes e_α for $\alpha \in A^* \setminus \{\epsilon\}$. □

3.6. **Topological Universality of Complex Bordism.**

Theorem 3.6.1. *Let E be a complex oriented A-spectrum. Then there exists a unique homotopy ring homomorphism $\mathbf{MU}_A \to E$ that sends the complex orientation of $\mathbf{MU}_A$ from Example 3.1.10 to the complex orientation of E.*

Phrased differently, the functor $\mathrm{CAlg}(\mathrm{h}(\mathrm{Sp}_A)) \to \mathrm{Set}$ sending a commutative homotopy ring spectrum E to the set of complex orientations of E is co-represented by the complex bordism spectrum $\mathbf{MU}_A$ together with the *universal complex orienation* $x^{\mathbf{MU}}(\epsilon) \in \mathbf{MU}_A^2(\mathbb{CP}_A^\infty, \mathbb{CP}(\epsilon))$. The *universal complex orientation* is uniquely determined by the Thom class $t^{\mathbf{MU}}(\gamma_1) \in \mathbf{MU}_A^2(\mathbb{CP}_A^\infty)^{\gamma_1}$ of the universal complex line bundle constructed in Example 3.1.10.

Proof. Let $\{V_n\}$ be a complete flag for A. Then, the transition maps of the telescoping formula for $\mathbf{MU}_A$ from Corollary 2.9.18 correspond under the Thom isomorphism in E-cohomology to the split surjections from Corollary 3.5.5. Thus, precomposition by the maps

$$t^{\mathbf{MU}}(\gamma_n)\cdot(t^{\mathbf{MU}}(V_n))^{-1} \overset{(50)}{=} \frac{t_d^U}{t^U(V_n)} : \Sigma^\infty \mathrm{B}_A(U(d))^{\gamma_d} \otimes S^{-V_n} \to \mathbf{MU}_A$$

induce a bijection

$$[\mathbf{MU}_A, E]^A \xrightarrow{\cong} \varprojlim_n [\Sigma^\infty \mathrm{B}_A(U(n))^{\gamma_n} \otimes S^{-V_n}, E]^A, \tag{80}$$

because of $\lim^1$-vanishing by surjectivity of the transition maps.

If $f : \mathbf{MU}_A \to E$ is orientation preserving, then the coordinate $y^{\mathbf{MU}}(\epsilon)$ is sent to $y^E(\epsilon)$. If f is moreover homotopy multiplicative, then $f_*(y(\epsilon)^{\times d}) = y(\epsilon)^{\times d}$ and therefore $f_*(t^{\mathbf{MU}}(\gamma_d)) = t^E(\gamma_d)$ by Construction 3.5.6. Hence, if f is additionally homotopy unital, then f sends $t^{\mathbf{MU}}(\gamma_n)\cdot(t^{\mathbf{MU}}(V_n))^{-1}$ to $t^E(\gamma_n)\cdot(t^E(V_n))^{-1}$. By Equation (80), the images of $t^{\mathbf{MU}}(\gamma_n)\cdot(t^{\mathbf{MU}}(V_n))^{-1}$ running through all $n \in \mathbb{N}_0$ already uniquely determine f. This establishes uniqueness.

For existence, we prove that the unique morphism of spectra $f : \mathbf{MU}_A \to E$ sending the product $t^{\mathbf{MU}}(\gamma_n)\cdot(t^{\mathbf{MU}}(V_n))^{-1}$ to the product $t^E(\gamma_n)\cdot(t^{\gamma_n}(V_n))^{-1}$ for all $n \in \mathbb{N}_0$ is an orientation preserving homotopy ring morphism. The morphism f is homotopy unital, by its defining condition for $n = 0$. To see that f is homotopy multiplicative, we investigate the set $[\mathbf{MU}_A \otimes \mathbf{MU}_A, E]^G$. To this end, we write $\mathbf{MU}_A \otimes \mathbf{MU}_A$ as an $\mathbb{N}_0$-indexed colimit by using our flag on both tensor factors and restricting the $\mathbb{N}_0 \times \mathbb{N}_0$-indexed colimit along the diagonal. To check that the transition maps of this colimit induce surjections in E-cohomology we apply the Thom isomorphism and the Künneth formula in E-cohomology[18] to reduce to surjectivity of the split surjection from Corollary 3.5.5. By $\lim^1$-vanishing, we conclude that a map $[\mathbf{MU}_A \otimes \mathbf{MU}_A, E]^G$ is uniquely determined by where binary products of the classes $t^{\mathbf{MU}}(\gamma_n)\cdot(t^{\mathbf{MU}}(V_n))^{-1}$ are sent. As the Thom classes are multiplicative with respect to direct sums of bundles in both $\mathbf{MU}_A$ cohomology, as well as E-cohomology, we conclude that f is homotopy multiplicative. Now, it follows from the defining property of f that $t^{\mathbf{MU}}(\gamma_n)$ is sent to $t^E(\gamma_n)$. In particular, f is orientation preserving. □

3.6.1. *Homology of the Complex Bordism Spectrum.* For a compact abelian Lie group A, the first computation of the oriented homology of the A-equivariant bordism spectrum $\mathbf{MU}_A$ appeared in work of Cole, Greenlees and Kriz [CGK02]. This computation contained an inaccuracy which was fixed in Julius Groenjes' Master thesis [Gro24].

Proposition 3.6.2. *Let E be a complex orientable A-spectrum. Then, the Thom isomorphism $E^A_*(\mathbb{CP}^\infty_A) \simeq E^A_*((\mathbb{CP}^\infty_A)^{\gamma_1} \otimes \mathbb{S}^{-2})$ together with the canonical A-map $(\mathbb{CP}^\infty_A)^{\gamma_1} \otimes \mathbb{S}^{-2} \to \mathbf{MU}_A$ induce an graded E_*-algebra isomorphism*

$$\left\{\mathrm{Sym}_*\left(E^A_*(\mathbb{CP}^\infty_A)\right)/(1-\vartheta(\epsilon))\right\}[\vartheta(\alpha)^{-1} : \alpha \in A^*] \xrightarrow{\cong} E^A_*(\mathbf{MU}_A).$$

Here, $\vartheta(\alpha)$ is the image of the unit under $E_*(* \xrightarrow{\alpha} \mathbb{CP}^\infty_A)_+$.

Proof. We choose a complete flag $\{V_n\}$ for A with $V_1 = \epsilon$ the trivial representation. Using this flag, we write $\mathbf{MU}_A$ as telescoping colimit as in Corollary 2.9.18. Via the Thom isomorphism, the E-homology of the n-th term of the telescope identifies with

$$E^A_*(\mathrm{B}_A(U(n))_+) \xrightarrow[\cong]{(74)} \{E^A_*\left(\mathbb{CP}^\infty_{A\,+}\right)^{\otimes n}\}_{\Sigma_n}.$$

Under this identification, the n-th stabilization map in the E-homology of the telescope computing $E^A_*(\mathbf{MU}_A)$ identifies with $\vartheta(\alpha_n) \otimes (-)$ for $\alpha_n \in A^*$ the orthogonal complement $V_n - V_{n-1}$. Now, the result follows from the standard description of the symmetric algebra and the telescoping formula for inversion of elements, because for each $\alpha \in A^*$ there are infinitely many $n \in \mathbb{N}$ with $\alpha = \alpha_n$. □

3.7. **Equivariant Quillen's Theorem.** In analogy to Quillen's computation [Qui69] of $\pi_* \mathbf{MU}$ it has long been conjectured, see [CGK00], that the homotopy groups of A-equivariant complex bordism coincide with the A-equivariant Lazard ring. This conjecture was resolved for compact abelian Lie groups by Markus Hausmann [Hau22].

Theorem 3.7.1. *Let A be a compactly metrizable abelian group. The graded ring homomorphism $k : L_A \to \pi^A_*(\mathbf{MU}_A)$ classifying the graded A-equivariant formal group law associated to the universal complex orientation is an isomorphism.*

Proof. We choose surjective morphisms $\varphi_n : A \to A_n$ of topological groups exhibiting $A \cong \varprojlim_n A_n$ as inverse limit of compact abelian Lie groups A_n. By Theorem 3.4.5, the graded ring

[18] The Künneth formula holds for E-cohomology of products of $\mathrm{B}_A(U(d))$'s by Equation (56)

homomorphisms $L_{A_n} \to L_A$ classifying the co-restriction of the universal A-equivariant formal group law induce an isomorphism $\operatorname{colim} L_{A_n} \to L_A$. The composite morphism

$$L_{A_n} \to L_A \xrightarrow{k} \pi_*^A(\mathbf{MU}_A) \tag{81}$$

classifies the co-restricted A_n-equivariant formal group law. The statement under consideration is known in the case of compact Lie groups, see [Hau22]. Hence, the map $L_{A_n} \to \pi_*^{A_n}(\mathbf{MU}_{A_n})$ classifying the formal group law associated to the universal complex orientation of $\mathbf{MU}_{A_n}$ is an isomorphism. The composite graded ring map

$$L_{A_n} \xrightarrow{\cong} \pi_*^{A_n}(\mathbf{MU}_{A_n}) \to \pi_*^{A_n}((\varphi_n)_*(\mathbf{MU}_A)) \simeq \pi_*^A(\mathbf{MU}_A) \tag{82}$$

classifies the formal group law associated to the pushforward complex orientation of $(\varphi_n)_*(\mathbf{MU}_A)$, because the canonical homotopy ring map $\mathbf{MU}_{A_n} \to (\varphi_n)_* \mathbf{MU}_A$ is orientation preserving. Thus, by Remark 3.3.7, the two maps $L_{A_n} \to \pi_*^A(\mathbf{MU}_A)$ from Equation (81) and Equation (82) agree. By Construction 2.9.6, the composite graded ring maps

$$\pi_*^{A_n}(\mathbf{MU}_{A_n}) \to \pi_*^{A_n}((\varphi_n)_*(\mathbf{MU}_A)) \simeq \pi_*^A(\mathbf{MU}_A)$$

induce a graded ring isomorphism $\operatorname{colim}_n \pi_*^{A_n}(\mathbf{MU}_{A_n}) \to \pi_*^A(\mathbf{MU}_A)$. So, the composite graded ring homomorphisms

$$L_{A_n} \to L_A \xrightarrow{k} \pi_*^A(\mathbf{MU}_A)$$

induce a graded ring isomorphism $\operatorname{colim}_n L_{A_n} \xrightarrow{\cong} \pi_*^A(\mathbf{MU}_A)$, which factors as

$$\operatorname{colim}_n L_{A_n} \to L_A \xrightarrow{k} \pi_*^A(\mathbf{MU}_A).$$

Because $\operatorname{colim}_n L_{A_n} \to L_A$ is an isomorphism, the homomorphism $k : L_A \to \pi_*^A(\mathbf{MU}_A)$ is an isomorphism, as well. □

4. Factorization of the HKR-Character Map

Recall that for any compactly metrizable abelian group A, we constructed a symmetric monoidal restriction functor $\operatorname{Res}_A : \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_A$ and similarly in the unstable set up. The underlying spectrum functor $\operatorname{Res}_e : \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}$ admits a lax monoidal right adjoint $(-)^b : \mathrm{Sp} \to \mathrm{Sp}_{\mathrm{gl}}$. If E is a complex oriented spectrum, then the *Borel spectrum* $E^{bA} := \operatorname{Res}_A(E^b)$ is a complex orientable A-spectrum, see Section 4.2.

For any compact Lie group G, the functor $\operatorname{Res}_G : \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_G$ admits a left adjoint $(-)_{/\!/G}$, such that $\operatorname{Res}_e(X /\!/ G) \simeq X_{hG}$ computes the homotopy orbits of $X \in \mathrm{Sp}_G$.

In a stable category we denote by $[-,-]_*$ the negatively graded homotopy groups of the mapping spectrum, i.e.

$$[\Sigma_+^\infty X, E]_* := \pi_{-*} \operatorname{map}_{\mathrm{Sp}}(\Sigma_+^\infty X, E) =: E^*(X),$$

where X is a space. In this Section 4, the expression $E^*(X)$ is our notation for unreduced E-cohomology, whenever $E \in \mathrm{Sp}$.

Our main agenda of this section is interpreting/recalling all expressions appearing in the HKR-character map from [HKR00]. Afterwards, we prove the following factorization of the HKR-character map:

Theorem 4.0.1. *Let E be a p-local height n Lubin-Tate spectrum*[19] *and $X \in \mathcal{S}_G$ a compact G-space, where G is a finite group. Let the topological group $\mathbb{Z}_p^n$ denote the n-fold product of the*

[19] Slightly more generally, E can be chosen to be any homotopy ring spectrum for which the HKR-character map was originally defined for in [HKR00]

p-adic integers. Then, the HKR-character map from [HKR00] *makes the diagram*

$$\begin{array}{ccccc}
E^*(X_{hG}) & \xrightarrow{\quad HKR \quad} & & & L^*(E)\otimes_{E^*} E^*(\mathrm{Fix}_{\mathbb{Z}_p^n}(G,X)_{hG}) \\
\uparrow\cong & & & & \downarrow\cong \\
[\Sigma^\infty_+ X_{hG}, E]_* & & & & [\Sigma^\infty_+ \mathrm{Fix}_{\mathbb{Z}_p^n}(G,X)_{hG}, \Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})]_* \\
\uparrow\cong & & & & \uparrow\cong \\
[\Sigma^\infty_+ X_{/\!/G}, E^b]_* & \xrightarrow{\mathrm{Res}_{\mathbb{Z}_p^n}} & [\mathrm{Res}_{\mathbb{Z}_p^n}\Sigma^\infty_+ X_{/\!/G}, E^{b\mathbb{Z}_p^n}]_* & \xrightarrow{\Phi^{\mathbb{Z}_p^n}} & [\Phi^{\mathbb{Z}_p^n}(\mathrm{Res}_{\mathbb{Z}_p^n}(\Sigma^\infty_+ X_{/\!/G})), \Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})]_*
\end{array}$$

commute. While the left vertical isomorphisms follow from adjunction, the lower right vertical isomorphism is induced by the $\mathbb{Z}_p^n$-fixed-point-G-orbit interchange map

$$I : \mathrm{Fix}_{\mathbb{Z}_p^n}(G,X)_{hG} \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}\mathbb{Z}_p^n, X_{/\!/G}) \underset{\simeq}{\xrightarrow{2.7.5}} (\mathrm{Res}_{\mathbb{Z}_p^n}(X_{/\!/G}))^{\mathbb{Z}_p^n}$$

from Proposition 4.1.7. In Section 4.3, we provide a preferred graded E^*-algebra isomorphism

$$\Phi^{\mathbb{Z}_p^n}_{-*}(E^{b\mathbb{Z}_p^n}) \cong L(E^*), \tag{83}$$

where $L(E^*)$ is the rational E^*-algebra from [HKR00]. *The upper right vertical map in the diagram is induced by the ring map $E \to \Phi^{\mathbb{Z}_p^n}(E^{b\mathbb{Z}_p^n})$. It is an isomorphism, because $L(E^*)$ is flat over E^*.*

Remark 4.0.2. Because $L(E^*)$ is rational, see [HKR00, Proposition 6.5.], we have a preferred equivalence

$$L(E^*)\otimes_{E^*} E^*(\mathrm{Fix}_{\mathbb{Z}_p^n}(G,X)_{hG}) \cong L(E^*)\otimes_{E^*}\left(E^*\left(\mathrm{Fix}_{\mathbb{Z}_p^n}(G,X)\right)\right)^G.$$

As Hopkins, Kuhn and Ravenel show in [HKR00], the HKR-character map induces an isomorphism

$$L(E^*)\otimes_{E^*} E^*(X_{hG}) \xrightarrow{\mathrm{HKR}} L(E^*)\otimes_{E^*}\left(E^*\left(\mathrm{Fix}_{\mathbb{Z}_p^n}(X,G)\right)\right)^G$$

after base-change. The left hand side is already interesting when X is a point (so that $X_{hG} \simeq BG$), while the right hand side $\mathrm{Fix}_{\mathbb{Z}_p^n}(*,G) = \hom_{\mathrm{Grp}}(\mathbb{Z}_p^n, G)$ becomes algebraic, in that case.

4.1. Formal Loop Spaces.

Definition 4.1.1. Let A be a profinite abelian group and A^* its Pontryagin dual. Let $X \in \mathcal{S}_{\mathrm{gl}}$ be a global space. Lurie [Lur19, Construction 3.4.3.] defines the *formal loop space* as the colimit of internal Hom-spaces

$$\mathcal{L}^{A^*}(X) := \operatorname*{colim}_{\Lambda\subseteq A^*} \underline{\mathrm{Hom}}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_0^*, X).$$

In particular, if surjective morphisms $A \to A_n$ exhibit the topological group A as an $\mathbb{N}^{\mathrm{op}}$-indexed limit of finite abelian groups A_n, then

$$\mathcal{L}^{A^*}(X) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A_n, X).$$

Observation 4.1.2. In the situation of Definition 4.1.1, we have a preferred equivalence

$$\mathrm{Res}_e\left(\mathcal{L}^{A^*}(X)\right) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A_n, X) = \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}A, X) \underset{\simeq}{\xrightarrow{2.7.5}} (\mathrm{Res}_A(X))^A \tag{84}$$

of spaces.

Definition 4.1.3. Let A be a profinite abelian group. Let G be a finite group and $X \in \mathcal{S}_G$ a G-space. From [HKR00] we recall the following G-space

$$\mathrm{Fix}_A(G,X) := \coprod_{\alpha \in \hom_{\mathrm{Grp}}(A,G)} X^{\mathrm{Im}(\alpha)}.$$

where $g \in G$ acts via $g \cdot x_\alpha = (gx)_{c_g \circ \alpha}$. Here $c_g : G \to G$ denotes conjugation by g.

The following discussion maybe thought of as a variant of [Rez14, Section 3.2. and 3.3.].

Construction 4.1.4. Let A be a metrizable profinite abelian group and G a finite group. We construct an *A-fixed-point-G-orbit interchange map*

$$I : \mathrm{Fix}_A(G,X)_{hG} \to \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!\!/G}) \tag{85}$$

naturally in the G-space X.

First, we assume that A is a finite abelian group. We write $C(\alpha)$ for the largest subgroup of G whose center contains the image of a homomorphism $\alpha : A \to G$. Observe that we have an equivalence of G-spaces

$$\coprod_{[\alpha] \in \hom_{\mathrm{Grp}}(A,G)/G} G \times_{C(\alpha)} X^{\mathrm{Im}(\alpha)} \to \mathrm{Fix}_A(G,X), \qquad [g,x] \mapsto (gx)_{c_g \circ \alpha},$$

inducing an equivalence

$$\coprod_{[\alpha] \in \hom_{\mathrm{Grp}}(A,G)/G} (X^{\mathrm{Im}(\alpha)})_{hC(\alpha)} \to \mathrm{Fix}_A(G,X)_{hG} \tag{86}$$

of spaces.

Consider the group homomorphism

$$i \times \alpha : \, C(\alpha) \times A \to G, \qquad (g,a) \mapsto g \cdot \alpha(a),$$

and note that the map

$$X^{\mathrm{Im}(\alpha)} \to (i \times \alpha)^* X$$

is $C(\alpha) \times A$ equivariant. Passing to global orbits yields a composite map of global spaces

$$(X^{\mathrm{Im}(\alpha)})_{/\!\!/C(\alpha)} \times \mathrm{B}_{\mathrm{gl}}A \to ((i \times \alpha)^* X)_{/\!\!/C(\alpha) \times A} \to X_{/\!\!/G}, \tag{87}$$

where the second map comes from the lax limit structure of global spaces, see [LNP25, Theorem 6.17.]. For example, when $X = *$ is the point the composite in Equation (87) is just

$$\mathrm{B}_{\mathrm{gl}}(i \times \alpha) : \, \mathrm{B}_{\mathrm{gl}}C(\alpha) \times \mathrm{B}_{\mathrm{gl}}A \to \mathrm{B}_{\mathrm{gl}}G.$$

The adjoint of the composite in Equation (87) is given by

$$(X^{\mathrm{Im}(\alpha)})_{/\!\!/C(\alpha)} \to \underline{\mathrm{Hom}}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!\!/G}). \tag{88}$$

Applying the underlying spectrum functor $\mathrm{Res}_e : \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}$ to the previous map yields

$$(X^{\mathrm{Im}(\alpha)})_{hC(\alpha)} \to \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!\!/G}), \tag{89}$$

which is the restriction of the interchange map I to the $[\alpha]$-factor.

If surjective morphisms $A \to A_n$ exhibit the topological group A as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $A \cong \varprojlim A_n$ of finite abelian groups A_n, we define the A-fixed-point-G-orbit interchange map by $I := \mathrm{colim}_n I_n$. Here I_n denotes the A_n-fixed-points-G-orbits interchange map.

Observation 4.1.5. In the situation of Construction 4.1.4, the composite

$$X^{\mathrm{Im}(\alpha)} \to (X^{\mathrm{Im}(\alpha)})_{hC(\alpha)} \xrightarrow{(89)} \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!\!/G})$$

sends a fixed point x to the composite

$$\mathrm{B}_{\mathrm{gl}}A \xrightarrow{\mathrm{B}_{\mathrm{gl}}(\alpha)} \mathrm{B}_{\mathrm{gl}}\,\mathrm{Im}(\alpha) \xrightarrow{x_{/\!\!/G}} X_{/\!\!/G},$$

where the latter map is obtained from the G-map $x : G/\,\mathrm{Im}(\alpha) \to X$ representing the $\mathrm{Im}(\alpha)$-fixed point x.

The A-fixed-point-G-orbit interchange map is compatible with inducing up the finite group G:

Remark 4.1.6. Let A be a metrizable profinite abelian group. Let G be a finite group and $X \in \mathcal{S}_H$ an H-space for $i : H \to G$ the inclusion of a subgroup. We define an H-map

$$\mathrm{Fix}_A(H, X) \to \mathrm{Res}^G_H(\mathrm{Fix}_A(G, \mathrm{Ind}^G_H(X)))$$

by

$$x_\alpha \mapsto (eH \times_H x)_{i\circ\alpha} \in \mathrm{Fix}_A(G, \mathrm{Ind}^G_H(X))$$

for $\alpha \in \hom_{\mathrm{Grp}}(A, H)$ and $x_\alpha \in X^{\mathrm{Im}(\alpha)}$. The adjoint G-map

$$\mathrm{Ind}^G_H(\mathrm{Fix}_A(H, X)) \longrightarrow \mathrm{Fix}_A(G, \mathrm{Ind}^G_H(X)) \tag{90}$$

is an equivalence. Moreover, the diagram

$$\begin{array}{ccc} \Big(\mathrm{Ind}^G_H(\mathrm{Fix}_A(H, X))\Big)_{hG} & \xrightarrow{\quad I \quad} & \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}A, \mathrm{Ind}^G_H(X)_{/\!\!/G}) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Fix}_A(H, X)_{hH} & \xrightarrow{\quad I \quad} & \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!\!/H}) \end{array}$$

has a canonical filler.

Proposition 4.1.7. *Let A be a metrizable profinite abelian group. Let G be a finite group and X a G-space. Then, the A-fixed-point-G-orbit interchange map*

$$\mathrm{Fix}_A(G, X)_{hG} \xrightarrow{I} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\mathrm{gl}})}(\mathrm{B}_{\mathrm{gl}}A, X_{/\!\!/G}) \underset{\simeq}{\xrightarrow{2.7.5}} (\mathrm{Res}_A(X_{/\!\!/G}))^A$$

is an equivalence, "interchanging A-fixed points with G-orbits".

Proof. By passage to colimits, it suffices to treat the case that A is a finite abelian group. Since both the domain and codomain preserve colimits in the variable X, it is enough to treat the case of orbits $X = G/H$. As the interchange map is compatible with induction (see Remark 4.1.6), it further suffices to restrict to the case that X is the point. In other words, we need to prove that the map

$$\hom_{\mathrm{Grp}}(A, G) \to \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A, \mathrm{B}_{\mathrm{gl}}G), \qquad \alpha \mapsto \mathrm{B}_{\mathrm{gl}}(\alpha)$$

induces an equivalence $\hom_{\mathrm{Grp}}(A, G)_{hG} \to \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}A, \mathrm{B}_{\mathrm{gl}}G)$ on homotopy orbits. A proof of this fact can be found in [Kö18, Proposition 2.5.]. □

4.2. **Orientations of Borel Spectra.**

Construction 4.2.1 (Orientation of Borel Spectra)**.** Let A be a compactly metrizable abelian group. Let $x(\epsilon) \in E^2(\mathbb{CP}^\infty, \mathbb{CP}(\epsilon))$ be a complex orientation of a commutative homotopy ring spectrum E. If we choose as representative of the class $x(\epsilon)$

$$x(\epsilon) : \mathrm{Res}_e\left(\Sigma^\infty \mathrm{B}_{\mathrm{gl}} U(1)\right) \to E \otimes S^2,$$

then its adjoint $\Sigma^\infty \mathrm{B}_{\mathrm{gl}} U(1) \to E^b \otimes S^2$ restricts to a map

$$x(\epsilon) : \Sigma^\infty \mathbb{CP}^\infty_A \to E^{bA} \otimes S^2$$

of A-spectra. This represents a well-defined cohomology class $x(\epsilon) \in (E^{bA})^2_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\epsilon))$.

Lemma 4.2.2. In situation of Construction 4.2.1, the class $x(\epsilon) \in (E^{bA})^2_A(\mathbb{CP}^\infty_A, \mathbb{CP}(\epsilon))$ is a complex orientation of the Borel spectrum E^{bA}.

Proof. Suppose surjective morphisms $A \to A_n$ exhibit the topological group A as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $A \cong \varprojlim_n A_n$ of abelian compact Lie groups A_n. Let $\alpha \in A^*$ be a character. By Theorem 2.1.4, we may choose $n \in \mathbb{N}$ such that $\alpha = \mathrm{Infl}^A_{A_n}(\alpha)$. The map representing

$$\mathrm{Res}_{\alpha\oplus\epsilon}(x(\alpha)) \in (E^{bA})^2_A(\mathbb{CP}(\alpha \oplus \epsilon), \mathbb{CP}(\epsilon))$$

factors through the homotopy ring homomorphism $\mathrm{Infl}^A_{A_n}(E^{bA_n} \otimes S^2) \to E^{bA} \otimes S^2$ as follows:

$$\Sigma^\infty \mathbb{CP}(\alpha \oplus \epsilon) \to \mathrm{Infl}^A_{A_n}\left(\Sigma^\infty \mathbb{CP}^\infty_{A_n}\right) \to \mathrm{Infl}^A_{A_n}(E^{bA_n} \otimes S^2) \to E^{bA} \otimes S^2.$$

Since inflation preserves representation-graded units, we may assume that A is a compact abelian Lie group.

As in Lemma 3.1.5, we identify $\mathbb{CP}(\alpha \oplus \epsilon)/\mathbb{CP}(\epsilon)$ with the representation sphere $S^{\alpha^{-1}}$. Via the adjunction $(-)_{hA} \dashv (-)^{bA}$, the Thom isomorphism in E-cohomology applied to the bundle $\alpha^{-1} \times_A EA$ over BA implies that multiplication by $\mathrm{Res}_{\alpha\oplus\epsilon}(x(\alpha))$ defines an isomorphism

$$[\Sigma^\infty S^{\alpha^{-1}}, E^{bA}]_* \to [\Sigma^\infty S^2, E^{bA}]_*.$$

Applying the same argument to any closed subgroup $B \leq A$ and the B-representation $\mathrm{Res}^A_B(\alpha^{-1})$ shows that multiplication by $\mathrm{Res}_{\alpha\oplus\epsilon}(x(\alpha))$ induces an isomorphism

$$\pi^B_*(E^{bA} \otimes S^{\alpha^{-1}}) \longrightarrow \pi^B_*(E^{bA} \otimes S^2)$$

on B-fixed homotopy groups. Because the functors $\pi^B_*(-)$ are jointly conservative, the class $x(\epsilon)$ satisfies Definition 3.1.4. □

4.2.1. *Euler Classes of Borel Spectra.* Let $x(\epsilon) \in E^2(\mathbb{CP}^\infty, \mathbb{CP}(\epsilon))$ be a complex orientation of a commutative homotopy ring spectrum and $y(\epsilon) \in E^2(\mathbb{CP}^\infty)$ the associated coordinate. For any character $\alpha \in A^*$ of a compactly metrizable abelian group A, the Euler class $e_\alpha \in \pi^A_*(E^{bA})$ associated to the complex orientation of the Borel A-spectrum E^{bA} is equal to the following construction:

Suppose surjective morphisms $A \to A_n$ exhibit A as an $\mathbb{N}^{\mathrm{op}}$-indexed inverse limit $A \cong \varprojlim_n A_n$ of compact abelian Lie groups. By Theorem 2.1.4, we may choose an $n \in \mathbb{N}^{\mathrm{op}}$ such that $\alpha = \mathrm{Infl}^A_{A_n}(\alpha)$. By definition, the Euler class e_α is the pullback of the coordinate associated to the complex orientation of E^{bA},

$$y(\epsilon) \in [\Sigma^\infty_+ \mathbb{CP}^\infty_A, E^{bA}]_*,$$

along the following composite map of A-spaces:

$$* \xrightarrow{\alpha} \mathrm{Infl}^A_{A_n} \mathbb{CP}^\infty_{A_n} \to \mathbb{CP}^\infty_A.$$

Thus, the map representing the Euler class e_α factors through the A_n-to-A inflation of the coordinate $y(\epsilon)$ of the Borel A_n-spectrum E^{bA_n}:

$$\Sigma^\infty_+ * \xrightarrow{\alpha} \Sigma^\infty_+ \operatorname{Infl}^A_{A_n} \mathbb{CP}^\infty_{A_n} \xrightarrow{\operatorname{Infl}^A_{A_n}(y(\epsilon))} \operatorname{Infl}^A_{A_n}(E^{bA_n}) \otimes S^2 \to E^{bA} \otimes S^2.$$

The map of A-spectra

$$\Sigma^\infty_+ * \xrightarrow{\alpha} \Sigma^\infty \operatorname{Infl}^A_{A_n} \mathbb{CP}^\infty_{A_n} \xrightarrow{\operatorname{Infl}^A_{A_n}(y(\epsilon))} \operatorname{Infl}^A_{A_n}(E^{bA_n}) \otimes S^2$$

is the A_n-to-A inflation of the map of A_n-spectra

$$\Sigma^\infty_+ * \xrightarrow{\alpha} \Sigma^\infty \mathbb{CP}^\infty_{A_n} \xrightarrow{y(\epsilon)} E^{bA_n} \otimes S^2.$$

Under the adjunction $(-)_{hA_n} \dashv (-)^{bA_n}$ the latter composite corresponds to the following composite map of spectra

$$\Sigma^\infty_+ \mathrm{B}A_n \xrightarrow{\mathrm{B}\alpha} \Sigma^\infty_+ \mathbb{CP}^\infty \xrightarrow{y(\epsilon)} E \otimes S^2.$$

Here, the map $\mathrm{B}\alpha : \mathrm{B}A_n \to \mathrm{B}U(1) = \mathbb{CP}^\infty$ is obtained by applying the classifying space functor $\mathrm{B}(-)$ to the map $\alpha : A_n \to U(1)$ of compact Lie groups.

In the current setup, for a topological group $A \cong \varprojlim_n A_n$, and a complex orientation $x(\epsilon)$ of a commutative homotopy ring spectrum E, we have proven the following Proposition:

Proposition 4.2.3 (Euler Classes of Oriented Borel Spectrum)**.** *Recall from Equation (45) that the inflations induce a graded ring isomorphism*

$$\operatorname*{colim}_{n\in\mathbb{N}} E^{-*}(\mathrm{B}A_n) \to \pi^A_*(E^{bA}). \tag{91}$$

For any character $\alpha \in A^*$ there is $n \in \mathbb{N}$ such that α factors through A_n. Then, the Euler class $e_\alpha \in \pi^A_{-2}(E^{bA})$ lifts to the class $c_1(\alpha) \in E^2(\mathrm{B}A_n)$, where the Chern class *$c_1(\alpha)$ is defined as the pullback of the coordinate $y(\epsilon) \in E^2(\mathbb{CP}^\infty)$ along*

$$\mathrm{B}\alpha : \mathrm{B}A_n \to \mathrm{B}U(1) = \mathbb{CP}^\infty. \tag{92}$$

Corollary 4.2.4 (Geometric Fixed Point of Oriented Borel Spectrum)**.** *Let $x(\epsilon)$ be a complex orientation of a homotpy ring spectrum E and suppose surjective morphisms $A \to A_n$ exhibit an topological group A as inverse limit $A \cong \varprojlim_n A_n$ of abelian compact Lie groups A_n. Then, the composite*

$$\operatorname*{colim}_{n\in\mathbb{N}} E^{-*}(\mathrm{B}A_n) \to \pi^A_*(E^{bA}) \to \Phi^*_A(E^{bA})$$

exhibits $\Phi^*_A(E^{bA})$ as the localization at the set

$$S := \{c_1(\alpha) \in E^2(\mathrm{B}A_n) \mid n \in \mathbb{N},\ \alpha \in A_n^* \setminus \{\epsilon\}\}$$

of Chern classes. Here $y(\epsilon) \in E^2(\mathbb{CP}^\infty)$ is the coordinate, which is obtained from the orientation $x(\epsilon)$ by forgetting the base point.

Proof. By Corollary 3.5.9, the map $\pi^A_*(E^{bA}) \to \Phi^*_A(E^{bA})$ is a localization at the following set of Euler classes

$$\{e_\alpha \in E^2_A \mid \alpha \in A^* \setminus \{\epsilon\}\}$$

associated to the complex orientation of the Borel spectrum E^{bA}. The result follows from Proposition 4.2.3. □

4.3. **The E^*-algebra $L^*(E)$.** In the following, we switch to the setup of Theorem 4.0.1, i.e. E is a p-local height n Lubin-Tate theory.

Notation 4.3.1. To simplify our notation we write $\Lambda := \mathbb{Z}_p^n$ and $\Lambda_r := (\mathbb{Z}/p^r\mathbb{Z})^n$.

Corollary 4.2.4 provides a preferred E^*-algebra isomorphism

$$L^{-*}(E) := S^{-1}\left(\operatorname*{colim}_{r\in\mathbb{N}} E^{-*}(\mathrm{B}\,\Lambda_r)\right) \longrightarrow \Phi_*^{\Lambda}(E^{b\Lambda}) \tag{93}$$

from the algebra $L^*(E)$, defined in [HKR00], to the geometric fixed points of the Borel theory $E^{b\Lambda} \in \mathrm{CAlg}(\mathrm{Sp}_\Lambda)$. Here, the set S is defined as the set of first Chern classes

$$S := \left\{c_1(\alpha) \in E^2(\mathrm{B}\Lambda_r) \mid r \in \mathbb{N},\, \alpha \in \Lambda_r^*,\ \alpha \neq \epsilon\right\}.$$

Remark 4.3.2. By [HKR00, Corollary 6.8.], the algebra $L^*(E)$ is rational and faithfully flat over $p^{-1}E^*$.

Remark 4.3.3. To the given formal group over $\pi_*(E)$ we associate a graded p-divisible group F_{p^∞}, by taking p-power-torsion points. The group of p^r-torsion points is given by $\mathrm{Spec}(E^*(\mathrm{B}\mathbb{Z}/p^r))$. The functor that associates to an E^*-algebra R the set of isomorphisms from the base change $R \otimes_{E^*} F_{p^\infty}$ to the constant p-divisible group $\mathbb{Q}_p^n/\mathbb{Z}_p^n$ is represented by $L^*(E)$, see [HKR00, Corollary 6.8.].

4.4. **Proof of the Factorization.** We are now in good shape to prove Theorem 4.0.1.

Notation 4.4.1. To simplify our notation we write $\Lambda := \mathbb{Z}_p^n$ and $\Lambda_r := (\mathbb{Z}/p^r\mathbb{Z})^n$ for $r \in \mathbb{N}$. Moreover, we suppress the notation of suspensions Σ_+^∞, in the non-equivariant, global, and Λ_r-equivariant setup.

Proof of Theorem 4.0.1. The diagram in Theorem 4.0.1 is compatible with Mayer Vietoris sequences and inductions along inclusions of finite groups Ind_H^G, see [HKR00, Section 6.4.]. Thus, we may assume that $X = * \in \mathcal{S}_G$ is the terminal G-space.

In the case that $X = * \in \mathcal{S}_G$ is the terminal G-space, the HKR-character map

$$\begin{aligned} E^*(\mathrm{B}G) &\xrightarrow{\mathrm{HKR}} \Phi_{-*}^{\Lambda}(E^{b\Lambda}) \otimes_{E^*} E^*(\hom_{\mathrm{Grp}}(\Lambda, G)_{hG}) \\ &\simeq \prod_{[\alpha]\in \hom_{\mathrm{Grp}}(\Lambda,G)/G} \Phi_{-*}^{\Lambda}(E^{b\Lambda}) \otimes_{E^*} E^*(\mathrm{B}C(\alpha)) \end{aligned}$$

is defined as follows:

To specify the HKR-character map on the factor of the conjugacy class of a homomorphism $\alpha : \Lambda \to G$, we choose $r \in \mathbb{N}$ such that α factors through $\alpha : \Lambda_r \to G$. The group homomorphism

$$\alpha \times i :\ \Lambda_r \times C(\alpha) \to G, \qquad (a, g) \mapsto \alpha(a) \cdot g$$

induce a map

$$\mathrm{B}(\alpha \times i) : \mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha) \to \mathrm{B}G$$

of spaces. Then, the $[\alpha]$-factor of the HKR-character map is given by the composite

$$\begin{aligned} E^*(\mathrm{B}G) &\xrightarrow{\mathrm{B}(\alpha\times i)^*} E^*(\mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha)) \xrightarrow[\cong]{\text{Künneth}} E^*(\mathrm{B}\Lambda_r) \otimes_{E^*} E^*(\mathrm{B}C(\alpha)) \\ &\to \Phi_{-*}^{\Lambda}(E^{b\Lambda}) \otimes_{E^*} E^*(\mathrm{B}C(\alpha)). \end{aligned} \tag{94}$$

Indeed, this is explained in [Sta13a, Example 4.1.7.].

Let us discuss the other composite in our Diagram 4.0.1

$$[\mathrm{B}G, E]_* \longrightarrow [\mathrm{Fix}_\Lambda(G,*)_{hG}, \Phi^\Lambda(E^{b\Lambda})]_* \cong \prod_{[\alpha]\in\hom_{\mathrm{Grp}}(\Lambda,G)/G} [\mathrm{B}C(\alpha), \Phi^\Lambda(E^{b\Lambda})]_*$$
$$\xrightarrow{\text{base-change}^{-1}} \prod_{[\alpha]\in\hom_{\mathrm{Grp}}(\Lambda,G)/G} \Phi^\Lambda_{-*}(E^{b\Lambda}) \otimes_{E^*} [\mathrm{B}C(\alpha), E]_*$$

which we claim to be equal to the HKR character map. We again discuss the factor corresponding to the conjugacy class of a group homomorphism $\alpha : \Lambda_r \to G$ for some $r \in \mathbb{N}$. By construction of the restriction functor $\mathrm{Res}_\Lambda : \mathrm{Sp}_{\mathrm{gl}} \to \mathcal{S}_\Lambda$ "as a colimit", the $[\alpha]$-factor of the previous composite factors as

$$[\mathrm{B}G, E]_* \simeq [\mathrm{B}_{\mathrm{gl}}G, E^b]_* \xrightarrow{\mathrm{Res}_{\Lambda_r}} [\mathrm{B}_{\Lambda_r}(G), E^{b\Lambda_r}]_* \xrightarrow{\Phi^{\Lambda_r}} [(\mathrm{B}_{\Lambda_r}(G))^{\Lambda_r}, \Phi^{\Lambda_r}(E^{b\Lambda_r})]_*$$
$$\xrightarrow{[\alpha]^*} [\mathrm{B}C(\alpha), \Phi^\Lambda(E^{b\Lambda})]_*. \tag{95}$$

Here we used that applying geometric fix points is compatible with colimits and inflations.

Consider the "Λ_r-equivariant G-principle bundle" $\bar{\alpha} : \mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha) \to \mathrm{B}_{\Lambda_r}(G)$, defined as the adjoint of the component inclusion

$$[\alpha] : \mathrm{B}C(\alpha) \to (\mathrm{B}_{\Lambda_r}(G))^{\Lambda_r} \simeq \mathrm{Map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, \mathrm{B}_{\mathrm{gl}}G) \tag{96}$$

at the conjugacy class of α. Recall that $\Phi^{\Lambda_r}(E^{b\Lambda_r})$ identifies with the categorical Λ_r-fixed points of the localization $S_r^{-1}E^{b\Lambda_r}$ of the Borel spectrum $E^{b\Lambda_r}$ at the set of Euler classes

$$S_r = \{e_\beta \in \pi_{-2}^{\Lambda_r}(E^{b\Lambda_r}) : \beta \in \Lambda_r^* \setminus \{\epsilon\}\}.$$

The previous composite in Equation (95) factors as

$$[\mathrm{B}G, E]_* \simeq [\mathrm{B}_{\mathrm{gl}}G, E^b]_* \xrightarrow{\mathrm{Res}_{\Lambda_r}} [\mathrm{B}_{\Lambda_r}(G), E^{b\Lambda_r}]_* \to [\mathrm{B}_{\Lambda_r}(G), S_r^{-1}E^{b\Lambda_r}]_*$$
$$\xrightarrow{\bar{\alpha}^*} [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), S_r^{-1}E^{b\Lambda_r}] \cong [\mathrm{B}C(\alpha), \Phi^\Lambda(E^{b\Lambda})]_*.$$

Now, we can invert the order of precomposition and postcomposition, to see that the composite in Equation (95) is homotopic to the following composite

$$[\mathrm{B}G, E]_* \simeq [\mathrm{B}_{\mathrm{gl}}G, E^b]_* \xrightarrow{\mathrm{Res}_{\Lambda_r}} [\mathrm{B}_{\Lambda_r}(G), E^{b\Lambda_r}]_* \xrightarrow{\bar{\alpha}^*} [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), E^{b\Lambda_r}]_*$$
$$\to [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), S_r^{-1}E^{b\Lambda_r}] \cong [\mathrm{B}C(\alpha), \Phi^\Lambda(E^{b\Lambda})]_*.$$

The adjunction $(-)_{h\Lambda_r} \dashv (-)^{b\Lambda_r}$ provides an isomorphism

$$[\mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha), E]_* \cong [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), E^{b\Lambda_r}]_*$$

and to conclude the proof it suffices to show the following two claims:

The first claim is that the composite

$$[\mathrm{B}G, E]_* \cong [\mathrm{B}_{\mathrm{gl}}G, E^b]_* \xrightarrow{\mathrm{Res}_{\Lambda_r}} [\mathrm{B}_{\Lambda_r}(G), E^{b\Lambda_r}]_*$$
$$\xrightarrow{\bar{\alpha}^*} [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), E^{b\Lambda_r}]_* \cong [\mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha), E]_* \tag{97}$$

agrees with precomposition along $\mathrm{B}(\alpha \times i)$. The second claim is that the composite

$$[\mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha), E]_* \cong [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), E^{b\Lambda_r}]_* \to [\mathrm{Infl}_1^{\Lambda_r} \mathrm{B}C(\alpha), S_r^{-1}E^{b\Lambda_r}]_* \cong [\mathrm{B}C(\alpha), \Phi^{\Lambda_r}(E^{b\Lambda_r})]_*$$

makes the diagram

$$\begin{array}{ccc} [\,\mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha), E]_* & \longrightarrow & [\mathrm{B}C(\alpha), \Phi^{\Lambda_r}(E^{b\Lambda_r})]_* \\ \Big\uparrow{\scriptstyle \text{Künneth}} & & \Big\uparrow{\scriptstyle \text{base-change}} \\ [\mathrm{B}\Lambda_r, E] \otimes_{E^*} [\mathrm{B}C(\alpha), E]_* & \longrightarrow & \Phi^{\Lambda_r}_{-*}(E^{b\Lambda_R}) \otimes_{E^*} [\mathrm{B}C(\alpha), E]_* \end{array}$$

commute.

To prove the second claim, we choose a morphisms of spectra $f : \mathrm{B}\Lambda_r \to E[k]$ and $g : \mathrm{B}C(\alpha) \to E[k']$ for integers $k, k' \in \mathbb{Z}$. Suppressing the notations of these shifts, it is sufficient to show that the diagram

$$\begin{array}{ccccc} \mathrm{B}C(\alpha) & \xrightarrow{\text{unit}^{\Lambda_r}} & \left((\mathrm{B}\Lambda_r \times \mathrm{B}C(\alpha))^{b\Lambda_r}\right)^{\Lambda_r} & \xrightarrow{(f^{b\Lambda_r}\cdot g^{b\Lambda_r})^{\Lambda_r}} & \left(E^{b\Lambda_r}\right)^{\Lambda_r} \\ \Big\downarrow{\scriptstyle f\otimes g} & & & & \Big\downarrow{\scriptstyle \simeq} \\ E \otimes E^{B\Lambda_r} & \longrightarrow & E^{B\Lambda_r} \otimes E^{B\Lambda_r} & \xrightarrow{\text{mult}} & E^{B\Lambda_r} \end{array}$$

commutes, up to homotopy. This is indeed the case because the multiplication of the Borel spectrum $E^{b\Lambda_r}$ is induced by the multiplication of E under the adjunction $(-)_{h\Lambda_r} \dashv (-)^{b\Lambda_r}$.

To prove the first claim, we express both adjunction isomorphism in Equation (97) via co-unit and units. Doing this carefully, we conclude that it is sufficient to show that the following diagram of spaces

$$\begin{array}{ccc} \mathrm{B}C(\alpha) \times \mathrm{B}\Lambda_r & \xrightarrow{\mathrm{B}(i\times\alpha)} & \mathrm{B}G \\ \Big\downarrow{\scriptstyle \bar{\alpha}_{h\Lambda_r}} & & \Big\uparrow{\scriptstyle \text{counit}} \\ (\mathrm{B}_{\Lambda_r}(G))_{h\Lambda_r} & \xrightarrow[(\mathrm{Res}_{\Lambda_r}(u))_{h\Lambda_r}]{} & ((\mathrm{B}G)^{b\Lambda_r})_{h\Lambda_r} \end{array}$$

commutes. Here, and in what follows, u shall denote the unit of the adjunction $\mathrm{Res}_e(-) \dashv (-)^b$ between spaces and global spaces.

We define a map of global spaces $\tilde{\mathrm{B}}(i \times \alpha)$ as the composite

$$\mathrm{const}(\mathrm{B}C(\alpha)) \times \mathrm{B}_{\mathrm{gl}}\Lambda_r \xrightarrow{\text{counit}\times \mathrm{B}_{\mathrm{gl}}\Lambda_r} \mathrm{B}_{\mathrm{gl}}C(\alpha) \times \Lambda_r \xrightarrow{\mathrm{B}_{\mathrm{gl}}(i\times\alpha)} \mathrm{B}_{\mathrm{gl}}G,$$

employing the counit of the adjunction $\mathrm{const} \dashv \mathrm{Res}_e(-)$. Since $\mathrm{const} : \mathcal{S} \to \mathrm{Sp}_{\mathrm{gl}}$ is fully faithful, the map $\mathrm{Res}_e(\tilde{\mathrm{B}}(i \times \alpha))$ identifies with $\mathrm{B}(i \times \alpha) : \mathrm{B}C(\alpha) \times \mathrm{B}\Lambda_r \to \mathrm{B}G$. We conclude that under the adjunction $\mathrm{Res}_e(-) \dashv (-)^b$ the adjoint morphism to the composite

$$\mathrm{const}(\mathrm{B}C(\alpha)) \times \mathrm{B}_{\mathrm{gl}}\Lambda_r \xrightarrow{\mathrm{B}(i\times\alpha)} \mathrm{B}_{\mathrm{gl}}G \xrightarrow{u} (\mathrm{B}G)^b$$

is given by $\mathrm{B}(i \times \alpha)$.

On top of that, the outer square of the following diagram

$$\begin{array}{ccccc} \mathrm{const}(\mathrm{B}C(\alpha)) \times \mathrm{B}_{\mathrm{gl}}\Lambda_r & \xrightarrow{\tilde{\mathrm{B}}(i\times a)} & \mathrm{B}_{\mathrm{gl}}G & \xrightarrow{u} & \mathrm{B}G^b \\ \Big\downarrow{\scriptstyle \bar{\alpha}_{/\!/\Lambda_r}} & \nearrow{\scriptstyle \text{counit}} & & & \Big\uparrow{\scriptstyle \text{counit}} \\ (\mathrm{B}_{\Lambda_r}(G))_{/\!/\Lambda_r} & & \xrightarrow[\mathrm{Res}_{\Lambda_r}(u)_{/\!/\Lambda_r}]{} & & \mathrm{Res}_{\Lambda_r}(\mathrm{B}G^b)_{/\!/\Lambda_r} \end{array}$$

corresponds, under the adjunction $\mathrm{Res}_e(-) \dashv (-)^b$, to the previous diagram. Hence, it suffices to show that the latter diagram commutes.

The right-hand part commutes by naturality of the counit of the adjunction $(-)_{/\!/\Lambda_r} \dashv \mathrm{Res}_\Lambda$. Hence, it is enough to prove that the upper-left triangle commutes. Under the adjunction

$$(-)_{/\!/\Lambda_r} \circ \mathrm{Infl}_1^{\Lambda^r} : \mathcal{S} \rightleftarrows \mathcal{S}_{\mathrm{gl}} : \mathrm{map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, -)$$

the composite map

$$\mathrm{counit} \circ \bar{\alpha}_{/\!/\Lambda_r} : \ \mathrm{const}(\mathrm{B}C(\alpha)) \times \mathrm{B}_{\mathrm{gl}}\Lambda_r \to \mathrm{B}_{\mathrm{gl}}G$$

of global spaces is adjoint to the map $[\alpha] : \mathrm{B}C(\alpha) \to \mathrm{map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, \mathrm{B}_{\mathrm{gl}}G)$ of spaces.

By construction of $[\alpha] : \mathrm{B}C(\alpha) \to \mathrm{map}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, \mathrm{B}_{\mathrm{gl}}G)$, the morphism

$$\tilde{\alpha} : \mathrm{B}_{\mathrm{gl}}C(\alpha) \to \underline{\mathrm{Hom}}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, \mathrm{B}_{\mathrm{gl}}G),$$

defined as the adjoint morphism of $\mathrm{B}_{\mathrm{gl}}(i \times \alpha) : \mathrm{B}_{\mathrm{gl}}C(\alpha) \times \Lambda_r \to \mathrm{B}_{\mathrm{gl}}G$, satisfies $\mathrm{Res}_e(\tilde{\alpha}) \simeq [\alpha]$. We conclude that under the adjunction $\mathrm{const}(-) \dashv \mathrm{Res}_e$ the composite

$$\mathrm{const}(\mathrm{B}C(\alpha)) \xrightarrow{\mathrm{counit}} \mathrm{B}_{\mathrm{gl}}C(\alpha) \xrightarrow{\tilde{\alpha}} \underline{\mathrm{Hom}}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, \mathrm{B}_{\mathrm{gl}}G)$$

corresponds to $[\alpha]$. Consequently, under the adjunction

$$((-) \times \mathrm{B}_{\mathrm{gl}}\Lambda_r) \circ \mathrm{const}(-) : \ \mathcal{S} \rightleftarrows \mathcal{S}_{\mathrm{gl}} : \mathrm{Res}_e \circ \underline{\mathrm{Hom}}_{\mathcal{S}_{\mathrm{gl}}}(\mathrm{B}_{\mathrm{gl}}\Lambda_r, -)$$

the morphism $[\alpha]$ corresponds to $\tilde{\mathrm{B}}(i \times \alpha)$.

To provide a homotopy between the map $\tilde{\mathrm{B}}(i \times \alpha)$ and the composite map $\mathrm{counit} \circ \bar{\alpha}_{/\!/\Lambda_r}$, it suffices to provide a homotopy between the two composite functors

$$(-)_{/\!/\Lambda_r} \circ \mathrm{Infl}_1^{\Lambda^r} \qquad \text{and} \ \ ((-) \times \mathrm{B}_{\mathrm{gl}}\Lambda_r) \circ \mathrm{const}(-),$$

because then the two maps are adjoint morphisms to $[\alpha]$ under homotopic functors. Each of these two composite functors preserves small colimits and sends the point to $\mathrm{B}_{\mathrm{gl}}\Lambda_r$. □

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