

SETUP OF MOTIVIC HOMOTOPY THEORY

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1. INTRODUCTION

Motivic homotopy theory aims to be to algebraic varieties what homotopy theory is to manifolds. The interaction between motivic and “ordinary” stable homotopy theory has proven to be remarkably fruitful, offering new tools to compute the homotopy groups of spheres.

In this first talk, we introduce the main objects of interest—spaces, spheres, stable stems, homotopy sheaves, and Thom spaces/spectra—in the world of motivic homotopy theory.

2. DEFINITION OF THE MOTIVIC CATEGORIES

What we morally want to do is take the category of schemes and make the affine line $\mathbb{A}^1$ contractible. In other words, we would like to universally (and homotopy coherently) invert the projection

$$X \times \mathbb{A}^1 \longrightarrow X$$

for all (sufficiently nice) schemes X .

However, the category of schemes does not admit all small colimits, so there is no controlled way to universally invert a collection of morphisms within this category. To address this, we replace the category of schemes with a sheaf category on finitely presented smooth schemes.

For this construction, we want a subcanonical Grothendieck topology on finitely presented smooth schemes. By analogy to homotopy theory we moreover would like a topology where

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a presheaf satisfies descent if and only if it satisfies some sort of excision property—that is, sends excisable squares to pullback squares. A basic example of such a topology is the Zariski topology.

Unfortunately, for the applications we have in mind,¹ the Zariski topology admits too few covers². We would prefer a topology that behaves more like the Euclidean topology—that is, one with finer coverings. The étale topology fits this intuition, but it is not generated by an excision property and suffers from certain drawbacks (such as infinite homotopical dimension and failure of descent for algebraic K -theory).

Instead, we will work with the *Nisnevich topology*, which lies between the Zariski and étale topologies.

Since we aim to obtain homotopy-invariant notions, our sheaves will take values in the ∞ -category of spaces $\mathcal{Spc}$. Throughout this seminar, we will use the language of ∞ -categories, which we will simply refer to as *categories* from now on.

2.1. The Nisnevich Topology. Let S be a scheme. To keep the discussion simple, we assume that S is quasi-compact, quasi-separated, Noetherian, and of finite Krull dimension. The Noetherian and finite-dimensional assumptions are not strictly necessary, but dropping them requires some technical adjustments in the definitions that follow. The overall theory (and proofs) remains essentially the same.

Recall that a morphism of finite type $f: X \rightarrow S$ is *smooth* if and only if it is flat and the relative cotangent sheaf $\Omega_{X/S}$ is locally free of rank equal to the relative dimension of X/S .

Definition 2.1.1. We denote by Sm_S the category of smooth S -schemes of finite type. For morphisms we allow all morphisms of S -schemes.

The category Sm_S is small and admits finite limits.

A morphism of schemes $X \rightarrow Y$ is called *étale* if it is smooth of relative dimension 0.

An *étale cover* $\{p_i: U_i \rightarrow X\}$ is a family of étale morphisms that are jointly surjective on the underlying Zariski spectra. In other words, for every field k and every k -point $x: \mathrm{Spec}(k) \rightarrow X$, there exists (possibly after a finite field extension) a lift of x to some U_i :

$$\begin{array}{ccc} U_i & \longrightarrow & X \\ \uparrow & & \uparrow \\ \mathrm{Spec}(\tilde{k}) & \dashrightarrow & \mathrm{Spec}(k) \end{array}$$

A *Nisnevich cover* $\{p_i: U_i \rightarrow X\}$ is a family of étale morphisms such that for every field k and every k -point $x: \mathrm{Spec}(k) \rightarrow X$, there exists a lift of x to some U_i *without* extending the residue field:

$$\begin{array}{ccc} U_i & \longrightarrow & X \\ & \nearrow & \uparrow \\ & & \mathrm{Spec}(k) \end{array}$$

Hence, every Nisnevich cover is an étale cover.

A *Zariski cover* $\{p_i: U_i \rightarrow X\}$ is a jointly surjective family of open immersions. Since open immersions induce isomorphisms on residue fields, every Zariski cover is a Nisnevich cover.

To construct an étale cover that is *not* Nisnevich, consider a nontrivial separable field extension, for instance

$$\mathrm{Spec}(\mathbb{C}) \longrightarrow \mathrm{Spec}(\mathbb{R}).$$

Fact. Admitting étale descent is equivalent to satisfying both Nisnevich descent and finite étale descent (i.e., descent for finite étale covers).

¹For instance, homotopical purity does not hold for the Zariski topology.

²And therefore $\mathrm{Sch} \rightarrow \mathrm{Sh}^{\mathrm{Zar}}(\mathrm{Sch})$ preserves too few colimits

A presheaf $F: \mathbf{Sm}_S^{\text{op}} \rightarrow \mathcal{S}\text{pc}$ satisfies *Nisnevich descent* if and only if it satisfies *Nisnevich excision* in the following sense.

Definition 2.1.2. A pullback square in $\mathbf{Sm}_S$

$$\begin{array}{ccc} W & \xrightarrow{i'} & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

is called a *Nisnevich square* if

- i is an open immersion,
- p is étale, and
- the induced morphism

$$p^{-1}(X - U) \longrightarrow (X - U)$$

is an isomorphism, i.e. in the equivalent pullback diagram

$$\begin{array}{ccc} p^{-1}(X - U) & \longrightarrow & V \\ \downarrow & & \downarrow p \\ X - U & \xrightarrow{\text{reduced}} & X \end{array}$$

the left vertical map is an isomorphism.

Remark 2.1.3. Here $(X - U) \hookrightarrow X$ denotes the unique reduced closed immersion whose image is the complement of $i(U)$. Then $p^{-1}(X - U) \hookrightarrow V$ is the corresponding reduced closed immersion whose image is the preimage of the set $|X| - |U|$ under the set map $|p|$. Because $W \rightarrow V$ is an open immersion with image $|p|^{-1}(|U|)$, we see that $p^{-1}(X - U) \hookrightarrow V$ is the unique closed immersion with image the set $|V| - |W|$. Thus, assuming p is étale and i is an open immersion, the square above is Nisnevich precisely when

$$p: (V - W)_{\text{red}} \longrightarrow (X - U)_{\text{red}}$$

is an isomorphism.

Lemma 2.1.4. If $i: U \rightarrow X$ and $p: V \rightarrow X$ form a Nisnevich square, then $\{i, p\}$ is a Nisnevich cover of X .

Proof. Let $x \in X$. If $x \in U$, then x lifts along i . Otherwise, $x \in X - U$, and by definition of a Nisnevich square, x lifts along p . $\square$

In fact, the Nisnevich topology is generated by Nisnevich squares in the sense of *cd-structures*. We record the following key consequence without proof.

Theorem 2.1.5. Let $F: \mathbf{Sm}_S^{\text{op}} \rightarrow \mathcal{S}\text{pc}$ be a presheaf on smooth finite type S -schemes. Then F is a sheaf for the Nisnevich topology if and only if:

- $F(\emptyset) \simeq *$, and
- for every Nisnevich square $W = U \times_X V$, the canonical map

$$F(X) \longrightarrow F(U) \times_{F(W)} F(V)$$

is an equivalence.

It follows that the Yoneda embedding $\mathbf{Sm}_S \rightarrow \mathbf{Sh}^{\text{Nis}}(\mathbf{Sm}_S)$ preserves the initial object and sends Nisnevich squares to pushout squares.

Features of the Nisnevich topology.

- If $\dim(S) = d$, then the homotopical dimension of the topos $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_S)$ is at most d . (This fails for the étale topology, where this dimension can be infinite.) In particular, $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_S)$ is hypercomplete, and Postnikov towers converge.
- Because Nisnevich descent can be checked via Nisnevich excision, the subcategory of Nisnevich sheaves is closed under filtered colimits in the presheaf category. (This is very convenient in practice)
- The Nisnevich topology is subcanonical (since the fpqc topology—and hence the étale topology—is already subcanonical). In light of Theorem [Theorem 2.1.5](#), this means that a Nisnevich square is already a pushout in Sm_S . Morally, the Yoneda embedding $\mathrm{Sm}_S \hookrightarrow \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_S)$ preserves precisely those pushout squares that are Nisnevich—and not many more.

2.2. $\mathbb{A}^1$ -invariance.

Definition 2.2.1. A presheaf $X: \mathrm{Sm}_S^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is called $\mathbb{A}^1$ -invariant if for every $U \in \mathrm{Sm}_S$, the projection

$$U \times \mathbb{A}^1 \longrightarrow U$$

is sent to an equivalence

$$X(U) \xrightarrow{\simeq} X(U \times \mathbb{A}^1).$$

By general principles (see [\[Lur09, 5.5.4\]](#)), the inclusion of $\mathbb{A}^1$ -invariant presheaves into all presheaves admits a left adjoint, denoted $L_{\mathbb{A}^1}$. We can describe this left adjoint more concretely.

Definition 2.2.2. The *standard cosimplicial scheme* is the functor

$$\Delta^\bullet: \Delta \longrightarrow \mathrm{Sm}_S, \quad [n] \longmapsto \Delta_S^n \subseteq \mathbb{A}_S^{n+1},$$

where Δ_S^n is the closed subscheme cut out by the equation

$$T_0 + \cdots + T_n = 1,$$

and the structure maps are induced by partial projections and face inclusions.

Note that $\Delta_S^n \cong \mathbb{A}_S^n$ for all $n \geq 0$.

Theorem 2.2.3. Let $F \in \mathcal{P}(\mathrm{Sm}_S)$ be a presheaf. Then there is a unique equivalence under F

$$L_{\mathbb{A}^1} F \xrightarrow{\simeq} \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} F((-) \times \Delta_S^n). \quad (1)$$

An important consequence of [Equation \(1\)](#) is that the functor $L_{\mathbb{A}^1}$ preserves finite products.

Definition 2.2.4. A *motivic space* is an $\mathbb{A}^1$ -invariant Nisnevich sheaf

$$F: \mathrm{Sm}_S^{\mathrm{op}} \longrightarrow \mathrm{Spc}.$$

The category of motivic spaces is denoted

$$\mathrm{Spc}(S) \subseteq \mathcal{P}(\mathrm{Sm}_S).$$

Lemma 2.2.5. The inclusion $\mathrm{Spc}(S) \subseteq \mathcal{P}(\mathrm{Sm}_S)$ preserves filtered colimits and admits a left adjoint L_{Mot} , whose unit is given by

$$\mathrm{id} \longrightarrow L_{\mathrm{Mot}} := \mathrm{colim} \left(L_{\mathrm{Nis}} \rightarrow L_{\mathbb{A}^1} L_{\mathrm{Nis}} \rightarrow L_{\mathrm{Nis}} L_{\mathbb{A}^1} L_{\mathrm{Nis}} \rightarrow \cdots \right).$$

In particular, L_{Mot} preserves finite products, since filtered colimits, L_{Nis} , and $L_{\mathbb{A}^1}$ all do.

Proof. Because Nisnevich descent can be checked on Nisnevich squares, and filtered colimits commute with finite limits in $\mathcal{Spc}$, the subcategory of Nisnevich sheaves is closed under filtered colimits. By cofinality, the colimit above sends any presheaf F to a Nisnevich sheaf $L_{\text{Mot}}(F)$. Since the category of $\mathbb{A}^1$ -invariant presheaves is closed under all colimits, the same argument shows that $L_{\text{Mot}}(F)$ is also $\mathbb{A}^1$ -invariant. We conclude that L_{Mot} factors through $\mathcal{Spc}(S)$. As $\text{Map}(L_{\text{Mot}}F, X) \rightarrow \text{Map}(F, X)$ is an equivalence for any motivic space $X \in \mathcal{Spc}(S)$, this identifies L_{Mot} as the left adjoint to the inclusion. $\square$

Corollary 2.2.6. *The image of $L_{\text{Mot}} : \text{Sm}_S \rightarrow \mathcal{Spc}(S)$ consists of compact objects, which generate $\mathcal{Spc}(S)$ under small colimits. In particular, $\mathcal{Spc}(S)$ is compactly generated and $\mathcal{Spc}(S)^\omega$ is the smallest full subcategory of $\mathcal{Spc}(S)$ containing $L_{\text{Mot}}(\text{Sm}_S)$, which is closed under small colimits in $\mathcal{Spc}(S)$.*

Proof. L_{Mot} preserves compact objects, because its right adjoint preserves filtered colimits. $\square$

2.2.1. *Pointed motivic spaces.* We equip the category of pointed motivic spaces

$$\mathcal{Spc}(S)_* := \mathcal{Spc}(S)_*/ \simeq \mathcal{Spc}(S) \otimes_{\text{Pr}_L} \mathcal{Spc}_*$$

with the smash product $X \wedge Y := \text{cofib}(X \vee Y \rightarrow X \times Y)$. Then, $\mathcal{Spc}(S)_*$ is a presentably symmetric monoidal category, i.e. $\mathcal{Spc}(S)_*$ is presentable and the smash product commutes with small colimits in each variable.

2.2.2. *$\mathbb{A}^1$ -homotopy.* Given a map $H : X \times \mathbb{A}^1 \rightarrow Y$ in Sm_S and two S -points $a, b : S \rightarrow \mathbb{A}^1$. Then the maps $X \xrightarrow{(\text{id}_X, a)} X \times \mathbb{A}^1$ and $X \xrightarrow{(\text{id}_X, b)} X \times \mathbb{A}^1$ are inverses to the equivalence $\mathbb{A}^1 \times X \rightarrow X$ in $\mathcal{Spc}(S)$. We conclude

Lemma 2.2.7. The maps $H_a : X \rightarrow Y$ and $H_b : X \rightarrow Y$ are equivalent in $\mathcal{Spc}(S)$.

2.3. **Motivic Spheres.** The following are some of the most important examples of motivic spaces. We will perform a few computations to get a better feel for $\mathbb{A}^1$ -homotopy theory. To begin, consider a diagram of schemes

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ U & \xrightarrow{\text{open imm.}} & X \end{array}$$

We can compute the pullback of the above span as the open subscheme of V corresponding to the preimage of U under the underlying map of sets $V \rightarrow X$.

Let $\mathbb{G}_m$ denote the pointed S -scheme $(\mathbb{A}^1 \setminus 0, 1)$. By the above principle, the following squares

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^2 \setminus 0 \end{array} \quad \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow [1, x] \\ \mathbb{A}^1 & \xrightarrow{[x, 1]} & \mathbb{P}^1 \end{array}$$

are pullback squares. Moreover, these are Nisnevich squares, so that after contracting $\mathbb{A}^1$ we obtain pushout squares of motivic spaces:

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{\text{pr}_1} & \mathbb{G}_m \\ \downarrow \text{pr}_2 & & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{A}^2 \setminus 0 \end{array} \quad \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & * \\ \downarrow & & \downarrow [1, 1] \\ * & \xrightarrow{[1, 1]} & \mathbb{P}^1 \end{array}$$

Consider now the following diagram:

$$\begin{array}{ccccc}
 \mathbb{G}_m & \xleftarrow{\text{pr}_1} & \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{\text{pr}_2} & \mathbb{G} \\
 \uparrow = & & \uparrow & & \uparrow = \\
 \mathbb{G}_m & \xleftarrow{\text{id} \vee \text{col}} & \mathbb{G}_m \vee \mathbb{G}_m & \xrightarrow{\text{col} \vee \text{id}} & \mathbb{G}_m \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \xleftarrow{\quad} & * & \xrightarrow{\quad} & *
 \end{array}$$

Computing first the horizontal pushouts yields the cospan

$$\mathbb{A}^2 \setminus 0 \longleftarrow * \longrightarrow *,$$

whose pushout is $\mathbb{A}^2 \setminus 0$. Conversely, if we first take the vertical pushouts, we obtain the cospan

$$* \longleftarrow \mathbb{G}_m \wedge \mathbb{G}_m \longrightarrow *,$$

whose pushout is $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$. We have constructed equivalences of pointed motivic spaces

$$\Sigma(\mathbb{A}^1 \setminus 0) \simeq \mathbb{P}^1 \quad \text{and} \quad \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \simeq \mathbb{A}^2 \setminus 0,$$

where $\mathbb{A}^2 \setminus 0$ is pointed at $(1, 1)$ and $\mathbb{P}^1$ is pointed at $[1, 1]$.

Definition 2.3.1. For integers $d \geq j \geq 0$, we define the (d, j) -motivic sphere as the pointed motivic space

$$S^{d,j} := S^{d-j} \wedge \mathbb{G}_m^{\wedge j}.$$

Generalizing the above discussion via induction, we obtain the following basic examples of motivic spheres.

Proposition 2.3.2. *There are explicit equivalences of pointed motivic spaces*

$$S^{2,1} = S^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$$

and

$$S^{2n-1,n} = S^{n-1} \wedge \mathbb{G}_m^n \simeq \mathbb{A}^n \setminus \{0\}.$$

Applying $S^1 \wedge (-)$ to the latter equivalence yields

$$S^{2n,n} \simeq S^1 \wedge (\mathbb{A}^n \setminus 0) = \text{cofib}(\mathbb{A}^n \setminus 0 \rightarrow *) \simeq \text{cofib}(\mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n).$$

2.4. Base Change. We will treat the following results more thoroughly when we will discuss the motivic six functor formalism. For now it will be useful to have the following base change results at hand: If $f : T \rightarrow S$ is a morphism of schemes then precomposition by the base change functor

$$\text{Sm}_S \rightarrow \text{Sm}_T, \quad X \mapsto X \times_S T$$

defines a functor $f_* : \mathcal{P}(\text{Sm}_T) \rightarrow \mathcal{P}(\text{Sm}_S)$, which is right adjoint to the Yoneda extension

$$f^* : \mathcal{P}(\text{Sm}_S) \rightarrow \mathcal{P}(\text{Sm}_T), \quad (X \in \text{Sm}_S) \mapsto X \times_S T.$$

The functor f_* preserves $\mathbb{A}^1$ -invariant Nisnevich sheaves, so we will also denote by

$$f_* : \text{Spc}(T) \longrightarrow \text{Spc}(S)$$

its restriction to motivic spaces. The left adjoint of f_* is denoted f^* and satisfies $f^* \circ L_{\text{Mot}} \simeq L_{\text{Mot}} \circ f^*$. In particular, $f^*(X) = T \times_S X$ for all $X \in \text{Sm}_S$.

Lemma 2.4.1. The functor $f^* : \text{Spc}(S) \rightarrow \text{Spc}(T)$ preserves finite products.

Proof. Since the motivic localization L_{Mot} preserves finite products the statement follows from the analogous fact on the level of presheaves, see [Lur09, Proposition 6.1.5.2]. $\square$

If $f : T \rightarrow S$ is smooth, then $f^* : \mathcal{P}(\mathrm{Sm}_S) \rightarrow \mathcal{P}(\mathrm{Sm}_T)$ is given by precomposition by the forgetful functor $\mathrm{Sm}_T \rightarrow \mathrm{Sm}_S$. Consequently, the forgetful functor Yoneda extends to a further left adjoint $f_\# : \mathcal{Spc}(T) \rightarrow \mathcal{Spc}(S)$ of f^* .

Proposition 2.4.2 (Nisnevich Separation). *Let $\{f_i : U_i \rightarrow S\}$ be a Nisnevich cover of a scheme S .*

- (1) *The family of functors $\{f_i^* : \mathcal{P}(\mathrm{Sm}_S) \rightarrow \mathcal{P}(\mathrm{Sm}_{U_i})\}$ jointly detect Nisnevich (resp. motivic) equivalences;*
- (2) *The family of functors $\{f_i^* : \mathcal{Spc}(S) \rightarrow \mathcal{Spc}(U_i)\}$ is conservative.*

Proof. (2) follows from (1). (2) follows immediately from (1). For (1), let $h : F \rightarrow G$ be a morphism in $\mathcal{P}(\mathrm{Sm}_S)$ such that each pullback $f_i^*(h)$ is a Nisnevich (resp. motivic) equivalence. For any finite tuple $(i_1, \dots, i_n)$ write

$$f_{i_1, \dots, i_n} : U_{i_1} \times_S \cdots \times_S U_{i_n} \longrightarrow S$$

for the corresponding iterated fiber product. Each $f_{i_1, \dots, i_n}^*(h)$ is a Nisnevich (resp. motivic) equivalence as well.

Consider the augmented simplicial endofunctor $C_\bullet \rightarrow \mathrm{id}$ of $\mathrm{Fun}(\mathcal{P}(\mathrm{Sm}_S), \mathcal{P}(\mathrm{Sm}_S))$ whose n -th term is

$$C_n = \bigsqcup_{i_1, \dots, i_n} (f_{i_1, \dots, i_n})_\# f_{i_1, \dots, i_n}^*.$$

For every $X \in \mathrm{Sm}_S$ the colimit $\mathrm{colim}_\Delta C_\bullet(X) \rightarrow X$ is a Nisnevich covering sieve, hence for every presheaf F the map $\mathrm{colim}_\Delta C_\bullet(F) \rightarrow F$ is a Nisnevich equivalence. By the 2-out-of-3 property for Nisnevich (resp. motivic) equivalences, h is a Nisnevich (resp. motivic) equivalence if and only if $\mathrm{colim}_\Delta C_\bullet(h)$ is such.

Finally, Nisnevich (and motivic) equivalences are stable under colimits, so the assumption that each $f_i^*(h)$ is an equivalence implies $\mathrm{colim}_\Delta C_\bullet(h)$ is an equivalence, and thus h itself is an equivalence. This proves (1). $\square$

2.5. Digression: Motivic Thom Spaces. Motivic Thom spaces are essential in motivic homotopy theory, as they allow one to distinguish vector bundles over the same base even after contracting $\mathbb{A}^1$, and they serve as fundamental building blocks for cohomology theories such as algebraic cobordism.

Let $\mathrm{Vect}(\mathrm{Sm}_S)$ denote the ordinary category of vector bundles and vector bundle morphisms on smooth S -schemes. We view $\mathrm{Vect}(\mathrm{Sm}_S)$ as a symmetric monoidal category via the external product:

$$(E \rightarrow X) \times (E' \rightarrow Y) \longmapsto (E \times E' \rightarrow X \times Y).$$

We equip the arrow category $\mathrm{Fun}([1], \mathrm{Sm}_S)$ with the Day convolution symmetric monoidal structure, where the poset $[1]$ is symmetric monoidal via the minimum function. By inspection, the functor

$$\mathrm{Vect}(\mathrm{Sm}_S) \longrightarrow \mathrm{Fun}([1], \mathrm{Sm}_S), \quad (E \rightarrow X) \longmapsto ((E - X) \rightarrow X),$$

admits a natural lax symmetric monoidal structure.

The *Thom space functor* $\mathrm{Th}(-)$ is then defined as the following composite of lax symmetric monoidal functors:

$$\mathrm{Vect}(\mathrm{Sm}_S) \longrightarrow \mathrm{Fun}([1], \mathrm{Sm}_S) \longrightarrow \mathrm{Fun}([1], \mathcal{Spc}(S)) \xrightarrow{\mathrm{cofib}} \mathcal{Spc}(S)_*.$$

Intuitively, for a vector bundle $E \rightarrow X$, the motivic Thom space $\mathrm{Th}(E)$ is obtained from E by collapsing the complement of the zero section to a point, i.e.

$$\mathrm{Th}(E) \simeq E/(E - X),$$

just as in classical topology. This construction generalizes the familiar topological Thom space to the motivic world.

We denote by $\text{Vect}_S \subseteq \text{Vect}(\text{Sm}_S)$ the full subcategory spanned by vector bundles over S itself.

Lemma 2.5.1. The lax symmetric monoidal composite

$$\text{Vect}_S \hookrightarrow \text{Vect}(\text{Sm}_S) \xrightarrow{\text{Th}} \mathcal{Spc}(S)_*$$

is symmetric monoidal.

Proof sketch. We need to check that for any two vector bundles E, E' over S , the map

$$\text{Th}(E) \wedge \text{Th}(E') \longrightarrow \text{Th}(E \times E')$$

is a motivic equivalence. Choose a Nisnevich cover of S on which both E and E' trivialize. By applying [Proposition 2.4.2](#) to this cover, we can assume that both E and E' are trivial bundles. In this case, [Proposition 2.3.2](#) factors the map under inspection into equivalences:

$$(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\wedge n} \simeq (S^{2,1})^{\wedge n} \simeq S^{2n,n} \simeq \mathbb{A}^n/\mathbb{A}^n - \{0\}$$

□

In fact, $\text{Vect}(\text{Sm}_S) \xrightarrow{\text{Th}} \mathcal{Spc}(S)_*$ is symmetric monoidal as a slight adjustment of the previous proof shows.

2.6. Motivic Spectra. Just as the category of spectra is obtained from pointed spaces by freely inverting smashing with S^1 , we want to construct the category of motivic spectra from pointed motivic spaces by freely inverting smashing with $\mathbb{P}^1$:

Definition 2.6.1. The category $(\mathcal{SH}(S), \otimes)$ of motivic spectra is defined as the initial presentably symmetric monoidal category under pointed motivic spaces

$$\Sigma^\infty : \mathcal{Spc}(S)_* \longrightarrow \mathcal{SH}(S)$$

on which tensoring with the bigraded spheres $\mathbb{S}^{d,j} := \Sigma^\infty S^{d,j}$ defines an equivalence

$$\Sigma^{d,j} := \mathbb{S}^{d,j} \otimes (-) : \mathcal{SH}(S) \rightarrow \mathcal{SH}(S).$$

Notation 2.6.2. We write $\Omega^{d,j} := (\Sigma^{d,j})^{-1}$ and

$$\mathbb{S}^{-d,-j} := \Omega^{d,j}(\mathbb{S})$$

for all $d \geq j \geq 0$. k -fold shifts by $\mathbb{P}^1$, $\mathbb{G}_m$ and S^k , respectively, will be denoted by subscripts, e.g. $\Sigma_{\mathbb{P}^1}^k$ and $\Omega_{\mathbb{P}^1}^k$.

Let's unravel the definition:

Remark 2.6.3. Let $\mathcal{C}$ be presentably symmetric monoidal. Let $F : \mathcal{Spc}(S)_* \longrightarrow \mathcal{C}$ be a continuous symmetric monoidal functor which sends all motivic spheres $S^{d,j}$ to smash invertible objects. Then, there exists a unique symmetric monoidal continuous functor $\mathcal{SH}(S) \rightarrow \mathcal{C}$ making the diagram

$$\begin{array}{ccc} \mathcal{Spc}(S)_* & & \\ \downarrow \Sigma^\infty & \searrow F & \\ \mathcal{SH}(S) & \dashrightarrow & \mathcal{C} \end{array} \quad \begin{array}{c} \\ \\ \exists! \end{array}$$

commute.

Construction 2.6.4. For any morphism of base schemes $f : S \rightarrow T$ there is a unique symmetric monoidal continuous functor $f^* : \mathcal{SH}(T) \rightarrow \mathcal{SH}(S)$ equipped with an equivalence $f^* \Sigma^\infty \simeq \Sigma^\infty f^*$ of functors $\mathcal{Spc}(T)_* \rightarrow \mathcal{SH}(S)$. We denote the right adjoint of f^* by $f_* : \mathcal{SH}(S) \rightarrow \mathcal{SH}(T)$.

For proving a telescoping localization formula for motivic spectra we need the following Lemma:

Lemma 2.6.5. The cyclic permutation $(1, 2, 3) : (\mathbb{P}^1)^{\wedge 3} \rightarrow (\mathbb{P}^1)^{\wedge 3}$ is homotopic to the identity in $\mathcal{Spc}(S)_*$.

Proof. Under the explicit equivalence $(\mathbb{P}^1)^{\wedge 3} \simeq \text{cofib}(\mathbb{A}^3 \setminus 0 \rightarrow \mathbb{A}^3)$ from [Proposition 2.3.2](#) the swap action corresponds to the swap action of $\mathbb{A}^3$ (Because the motivic J -homomorphism is symmetric monoidal). It suffices to write $(1, 2, 3) : \mathbb{A}^3 \rightarrow \mathbb{A}^3$ as a composition of maps which each is $\mathbb{A}^1$ -homotopic to the identity via a homotopy that sends $\mathbb{A}^1 \times (\mathbb{A}^3 \setminus 0)$ to $\mathbb{A}^3 \setminus 0$.

To this end we factor the matrix encoding $(1, 2, 3)$ into a product of integer valued elementary matrices. For the elementary matrix $E(a)$ with unique nonzero off diagonal entry $a \in \mathbb{Z}$ the required $\mathbb{A}^1$ -homotpy is given by

$$\mathbb{A}^3 \times \mathbb{A}^1 \rightarrow \mathbb{A}^3, \quad (X, t) \mapsto E(ta)(X).$$

□

Theorem 2.6.6. We let $\mathcal{Spc}(S)_*[(\mathbb{P}^1)^{-1}]$ denote the colimit of the diagram

$$\mathcal{Spc}(S)_* \xrightarrow{\mathbb{P}^1 \wedge (-)} \mathcal{Spc}(S)_* \xrightarrow{\mathbb{P}^1 \wedge (-)} \mathcal{Spc}(S)_* \xrightarrow{\mathbb{P}^1 \wedge (-)} \dots$$

computed in the category of presentable categories and left adjoint functors Pr^L . Then this colimit is an idempotent algebra in left $\mathcal{Spc}(S)_*$ modules in Pr^L and the canonical continuous symmetric monoidal functor

$$\mathcal{Spc}(S)_*[(\mathbb{P}^1)^{-1}] \longrightarrow \mathcal{SH}(S)$$

is an equivalence of categories under $\mathcal{Spc}(S)_*$.

Proof. This is a categorification of the group completion theorem, see [\[Rob13\]](#). This theorem is applicable in our situation precisely because [Lemma 2.6.5](#) holds. □

Remark 2.6.7. Since Σ^∞ and $\otimes$ preserves colimits, the functor $\mathcal{SH}(S) \rightarrow \mathcal{SH}(S)$ given by

$$\Sigma_{S^1} := \Sigma^{1,1} = (\Sigma^\infty S^1) \otimes (-) = \text{colim}(0 \leftarrow (-) \rightarrow)$$

is equivalent to categorical suspension. We conclude that $\mathcal{SH}(S)$ is a stable category.

Corollary 2.6.8 (Motivic Spanier-Whitehead Category). *The category of motivic spectra $\mathcal{SH}(S)$ is compactly generated. Any compact motivic spectrum can be written as $\Omega_{\mathbb{P}^1}^n \Sigma^\infty X$ for some $n \in \mathbb{N}_0$ and some compact motivic space $X \in \mathcal{Spc}(S)_*$. Moreover, for two compact pointed motivic spaces $X, Y \in \mathcal{Spc}(S)_*$ the map*

$$\text{colim}_n \text{Map}_{\mathcal{Spc}(S)_*}((\mathbb{P}^1)^{\wedge n} \wedge X, (\mathbb{P}^1)^{\wedge n} \wedge Y) \longrightarrow \text{Map}_{\mathcal{SH}(S)}(\Sigma^\infty X, \Sigma^\infty Y)$$

is an equivalence.

Proof. By [\[Lur09, Chapter 5\]](#) the category of compact objects in the telescoping colimit

$$\mathcal{Spc}(S)_*[(\mathbb{P}^1)^{-1}] \simeq \mathcal{SH}(k)$$

is the telescoping colimit of the category of compact objects; here the latter colimit is computed in the category of small categories. The statement follows for the formula for objects and mapping spaces in filtered colimits of small colimits. □

Proposition 2.6.9. *If $f : S \rightarrow T$ is a smooth morphism of base schemes, then $f^* : \mathcal{SH}(T) \rightarrow \mathcal{SH}(S)$ admits a left adjoint $f_\# : \mathcal{Sh}(S) \rightarrow \mathcal{SH}(T)$ and the Beck-Chevalley transformations*

$$f_\# \Sigma^\infty \longrightarrow \Sigma^\infty f_\# \quad \text{and} \quad f_\# \Sigma_{\mathbb{P}^1} \longrightarrow \Sigma_{\mathbb{P}^1} f_\#$$

are equivalence of functors $\mathcal{Spc}(S)_ \rightarrow \mathcal{SH}(T)$ and $\mathcal{SH}(S) \rightarrow \mathcal{SH}(T)$, respectively.*

Proof. We claim that the canonical map

$$f_\# \circ (\mathbb{P}_S^1 \wedge (-)) \longrightarrow (\mathbb{P}_T^1 \wedge (-)) \circ f_\#$$

is an equivalence of functors $\mathcal{Spc}(S)_* \rightarrow \mathcal{Spc}(T)_*$. Indeed, for any $X \in \mathbf{Sm}_S$ the map

$$X \times_S (S \times_{\mathbb{Z}} \mathbb{P}^1) \longrightarrow X \times_T (T \times_{\mathbb{Z}} \mathbb{P}^1)$$

is an isomorphism in $\mathbf{Sm}_T$ and we might pass to colimits in X . In light of [Theorem 2.6.6](#) the result follows from the claim by very general arguments, e.g. see [\[Neu25, 2.6.16\]](#). Essentially, we define an $\mathbb{N}$ parametrized category $n \mapsto \mathcal{Spc}(S)_*$ with transition maps given by smashing with $\mathbb{P}_S^1$, and similarly for T . Now f^* provides a parametrized functor between these two parametrized categories. Because the Beck-Chevalley condition holds, we can pass to the parametrized left adjoint, pointwise given by $f_\#$. Finally, we take colimits in $\mathbb{N}$ to get our required functor $f_\# : \mathcal{SH}(S) \rightarrow \mathcal{SH}(T)$. $\square$

2.7. Example: Motivic Thom Spectrum. Motivic Thom spectra provide a bridge between algebraic vector bundles and stable motivic homotopy theory. Motivic Thom spectra are comparably understandable, but are not too far from the motivic sphere to yield a good approximation. The motivic Thom spectrum MGL_S *represents algebraic cobordism*—for our purposes this is by definition, though there are efforts to give a more geometric construction of algebraic cobordism.

We claim that the symmetric monoidal composite (see [Lemma 2.5.1](#))

$$\mathrm{Vect}_S \hookrightarrow \mathrm{Vect}(\mathbf{Sm}_S) \xrightarrow{\mathrm{Th}} \mathcal{Spc}(S)_* \xrightarrow{\Sigma^\infty} \mathcal{SH}(S)$$

sends all vector bundles over S to *smash-invertible* motivic spectra. Indeed, given a vector bundle E over S , we may choose a Nisnevich cover on which E trivializes. By the $\mathcal{SH}$ -analog of [Proposition 2.4.2](#) applied to that cover, we may assume that E is trivial. The Thom space of a trivial bundle over S is given by a motivic sphere, which we forced to be invertible in the construction of $\mathcal{SH}(S)$.

We view the ordinary groupoid Vect_S of vector bundles on S as a commutative monoid in spaces. From the claim, we deduce that the above functor factors through a unique symmetric monoidal functor

$$J : \mathrm{Vect}_S^{\mathrm{grp}} \longrightarrow \mathcal{SH}(S),$$

where $\mathrm{Vect}_S \rightarrow \mathrm{Vect}_S^{\mathrm{grp}}$ denotes the group-completion in $\mathbf{CMon}(\mathcal{Spc})$. This functor is called *motivic J -homomorphism*.

Construction 2.7.1. When $S = \mathrm{Spec}(R)$, then $\mathrm{Vect}_S^{\mathrm{grp}}$ is known as the algebraic K-theory $K(R)$ of R . When S is not affine, this is not true in general but [\[BH20, Bachmann-Hoyois, 16.1\]](#) show that J factors through the map $\mathrm{Vect}_S^{\mathrm{grp}} \rightarrow K(S)$ to algebraic (Thomason-Trobaugh) K-theory.³

Finally, restricting this J -homomorphism to virtual vector bundles of virtual rank 0 gives the motivic j -homomorphism

$$j : K(S)^0 \longrightarrow \mathcal{SH}(S),$$

where the map $K(S)^0 \rightarrow K(S)$ is the fiber in $\mathbf{CGrp}(\mathcal{Spc})$ of the rank map

$$\mathrm{rank} : K(S) \rightarrow \mathbb{Z},$$

³When I gave the talk, I ignored the difference between $\mathrm{Vect}_S^{\mathrm{grp}}$ and $K(S) = (\mathrm{Perf}(S)^{\mathrm{core}})^{\mathrm{grp}}$.

factoring the ordinary rank function $\text{rank} : \text{Vect}_S \rightarrow \mathbb{Z}$ through the algebraic K-theory.

In the seminar talk, I presented the following as the motivic Thom spectrum. But this definition is not quite big enough as I'll explain in a second.

Definition 2.7.2. We define the *preliminary motivic Thom spectrum* as the colimit of the j -homomorphism:

$$\text{Th}_S(j) := \text{colim}(j : K(S) \longrightarrow \mathcal{SH}(S)) \in \text{CAlg}(\mathcal{SH}(S)).$$

Note that the colimit of a symmetric monoidal functor naturally inherits a commutative algebra structure, by equipping the colimit functor with a symmetric monoidal structure with respect to Day convolution on the functor category.

By construction $\text{Th}_S(j)$ will have Thom isomorphisms for virtual vector bundles over the terminal S -scheme. But this is not enough to obtain Thom isomorphisms for virtual vector bundles over any smooth finite type S -scheme. This is then guaranteed by the following construction

Construction 2.7.3. Bachmann-Hoyois [BH20, Section 16] construct a functor

$$\text{Sm}_S \rightarrow \mathcal{SH}(S), \quad (f : X \rightarrow S) \mapsto f_{\#} \text{Th}_X(j)$$

manifesting the functoriality of the Thom spectrum construction. We define the *motivic Thom spectrum*

$$\text{MGL}_S := \text{colim}_{f \in \text{Sm}_S} f_{\#} \text{Th}_X(j).$$

Remark 2.7.4. Note that for a smooth S -scheme $f : X \rightarrow S$ and $E \in \text{Vect}_X$ we have

$$f_{\#} j(E) = f_{\#} \text{Th}(E) = \text{Th}(E)$$

in $\text{Spc}(S)_*$. Unraveling the definition of the motivic Thom spectrum we obtain

$$\begin{aligned} \text{MGL}_S &= \text{colim}_{f \in \text{Sm}_S} f_{\#} \text{Th}_X(j) = \text{colim}_{f \in \text{Sm}_S} f_{\#} \left(\text{colim}_{E \in K(X)^0} j(E) \right) \\ &\simeq \text{colim}_{f \in \text{Sm}_S, E \in K(X)^0} f_{\#} j(E) \\ &= \text{colim}_{f \in \text{Sm}_S, E \in K(X)^0} \text{Th}(E) \end{aligned}$$

where we define $\text{Th}(E) := f_{\#} j(E)$ for any *virtual* vector bundle $E \in K(X)^0$, c.f. [BH20, Theorem 16.13.].

Remark 2.7.5. One can show that the canonical map

$$\text{colim}_n \Omega_{\mathbb{P}^1}^n \Sigma^{\infty} \text{Th}(V_n) \longrightarrow \text{MGL}_S$$

is an equivalence, where V_n is the tautological bundle on the infinite Grassmannian

$$\text{Gr}(n)_S := \text{colim}_{k \rightarrow \infty} \text{Gr}(n, k)_S$$

of n -planes, see [BH20, 16.13].

Remark 2.7.6. MGL_S enhances to an $\mathbb{E}_{\infty}$ -algebra (even a normed motivic spectrum), see [BH20, Section 16.1]. To this end, one identifies MGL_S with the Sm_S^{op} -parametrized colimit of the Sm_S^{op} -parametrized j -homomorphism. The latter is the natural transformation

$$j : K(-)^0 \Rightarrow \mathcal{SH}(-)$$

of the two functors $\text{Sm}_S^{\text{op}} \rightarrow \text{Cat}$.

3. MOTIVIC STABLE STEMS

Definition 3.0.1. Let $X \in \mathcal{SH}(S)$ be a motivic spectrum and $d, j \in \mathbb{Z}$. We define the motivic stable homotopy groups as the abelian group of homotopy classes of maps

$$\pi_{d,j}(X) := [\mathbb{S}^{d,j}, X].$$

The i -th *motivic stable stem* is the graded abelian group

$$\pi_i(\mathbb{S})_j := \pi_{i-j, -j}(\mathbb{S}) = [\Sigma^\infty S^i, \Sigma^\infty \mathbb{G}_m^j].$$

The 0-th motivic stable stem $\pi_0(\mathbb{S})_*$ is a graded (non-commutative⁴) ring

Let us start by constructing an interesting element of $\pi_0(\mathbb{S})_*$. In topological homotopy theory the first interesting element of the stable stems is induced by the Hopf fibration $\eta : \mathbb{C}^2 \setminus 0 \simeq S^3 \rightarrow \mathbb{CP}^1 \simeq S^2$. The canonical map $\eta : \mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$ also exists in the world of smooth S -schemes. Under the explicit equivalence 2.3.2 this is sent to a map

$$\eta : \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \simeq S^{3,2} \rightarrow \Sigma \mathbb{G}_m \simeq S^{2,1}$$

of pointed motivic spaces. By suspending Σ^∞ we view η as an element

$$\eta \in \pi_0(\mathbb{S})_{-1}$$

of the stable stems.

Say $S = \text{Spec}(k)$ for k a field. Then we can construct a bunch of less interesting elements of the motivic stable stems as follows: For any $a \in k^*$, we obtain a map $*$ $\rightarrow \mathbb{G}_m$ of smooth k -schemes. This yields a map $S^0 \rightarrow \mathbb{G}_m$ of pointed motivic spaces, i.e. an element

$$[a] \in \pi_0(\mathbb{S})_1.$$

Definition 3.0.2. Let k be a field. The graded ring $K_*^{\text{MW}}(k)$ called *Milnor-Witt-K-Theory* of k is defined to be the quotient of the free non-commutative ring on generators $[a]$ in degree 1 for $a \in k^\times$ and a generator η in degree -1 , subject to the following relations

- (1) $\eta[a] = [a]\eta$,
- (2) $[a][1-a] = 0$ for $a \in k \setminus \{0, 1\}$, (*Steinberg relations*)
- (3) $[ab] = [a] + [b] + \eta[a][b]$,
- (4) $\eta(2 + \eta[-1]) = 0$.

Remark 3.0.3. The graded ring $K_*^{\text{M}}(k)$ called *Milnor K-theory* of k is defined as the quotient of $K_*^{\text{MW}}(k)$ by the central element η .

There is a natural map $K_*^{\text{M}}(k) \rightarrow K_*(k)$ to algebraic K -theory, which is an isomorphism in degree 0, 1 and 2, but not for larger degrees, in general.

Theorem 3.0.4 (Morel). *Let k be a field. The 0-th motivic stable stem $\pi_0(\mathbb{S})_*$ is via the elements constructed above isomorphic to Milnor-Witt K-theory $K_*^{\text{MW}}(k)$.*

4. STRICT $\mathbb{A}^1$ -INVARIANCE

Given a Nisnevich sheaf of abelian groups A then we can compute its n -fold delooping $K(A, n)$ in the topos $\text{Sh}^{\text{Nis}}(\text{Sm}_S)$. For any smooth scheme $X \in \text{Sm}_S$ the group $\pi_0 \text{Map}(X, K(A, n)) := [X, K(A, n)]$ computes the *sheaf cohomology group* $H_{\text{Nis}}^n(X; A)$ of X with coefficients in A . When $n = 1$, this remains true for a sheaf of not-necessarily-abelian groups G and then

$$[X, \text{BG}] = H_{\text{Nis}}^1(X; G)$$

is the group of G -torsors on X with respect to the Nisnevich topology.

⁴Morel proved that $\pi_0(\mathbb{S})_*$ is commutative up to multiplication by an element denoted ϵ .

If A is $\mathbb{A}^1$ -invariant then we can compute

$$H_{\text{Nis}}^0(X; A) \cong [X, A] \cong [L_{\mathbb{A}^1} X, A]_{\mathcal{Spc}(k)}$$

in the much nicer category $\mathcal{Spc}(k)$ of motivic spaces. Wouldn't it be nice if we could also compute

$$H^n(X; A) \cong [X, K(A, n)] \stackrel{?}{\cong} [L_{\mathbb{A}^1} X, K(A, n)]_{\mathcal{Spc}(k)}.$$

in the category $\mathcal{Spc}(k)$ of motivic spaces, i.e. wouldn't it be nice if

$$H^n(X \times \mathbb{A}^1; A) \rightarrow H^n(X; A)$$

was an isomorphism for all $n \in \mathbb{N}$. This is taken care of by the following (difficult) theorem:

Theorem 4.0.1 (Morel). *Let k be a perfect field and A a Nisnevich sheaf of abelian groups on Sm_k . Then, the following are equivalent*

- 1) *For $n = 0, 1$ and any $X \in \text{Sm}_S$ the map $H^n(X \times \mathbb{A}^1; A) \rightarrow H^n(X; A)$ is an isomorphism. (In this case A is called strongly $\mathbb{A}^1$ -invariant)*
- 2) *For all $n \in \mathbb{N}$ and any $X \in \text{Sm}_S$ the map $H^n(X \times \mathbb{A}^1; A) \rightarrow H^n(X; A)$ is an isomorphism. (In this case A is called strictly $\mathbb{A}^1$ -invariant)*

Example 4.0.2. For any $n \in \mathbb{N}_1$ consider the sheaf $\text{GL}_n : \text{Sm}_k^{\text{op}} \rightarrow \text{Grp}$ represented by the linear algebraic group GL_n . Since every Zariski GL_n -torsor is representable, every such torsor defines a Nisnevich sheaf, and therefor a torsor in the Nisnevich topology on Sm_k . Conversely, it is a theorem that any Nisnevich GL_n -torsor trivializes Zariski locally. We conclude that GL_n -torsor in the Zariski topology are canonically in bijection with GL_n -torsor in the Nisnevich topology:

$$H_{\text{Nis}}^1(-, \text{GL}_n) \cong H_{\text{Zar}}^1(-, \text{GL}_n).$$

In particular, for $\mathbb{G}_m = \text{GL}_1$ we obtain the Picard group

$$H_{\text{Nis}}^1(X; \mathbb{G}_m) \cong H_{\text{Zar}}^1(X; \mathbb{G}_m) \cong \text{Pic}(X)$$

as sheaf cohomology of any smooth k -scheme $X \in \text{Sm}_k$. It is not hard to see that $\mathbb{G}_m$ is $\mathbb{A}^1$ -invariant as we can compute the units $R[t]^\times = R^\times$ for any regular ring R . The sheaf $\text{Pic}(-) \simeq K(\mathbb{G}_m, 1)$ is also $\mathbb{A}^1$ -invariant on smooth k -schemes (this follows from a nontrivial result in commutative algebra). By the above theorem 4.0.1 of Morel, it then follows that $H_{\text{Nis}}^n(-, \mathbb{G}_m)$ is $\mathbb{A}^1$ -invariant for all $n \in \mathbb{N}$.

In fact, if k is a perfect field, and $X \in \text{Sm}_k$ is an *affine* smooth scheme, the canonical map

$$[X, \text{BGL}_n]_{\text{Sh}^{\text{Nis}}(\text{Sm}_k)} \rightarrow [X, L_{\mathbb{A}^1} \text{BGL}_n]_{\mathcal{Spc}(k)}$$

is an isomorphism. By the previous discussion, this group is equal to $H_{\text{Zar}}^1(X, \text{GL}_n)$ the set of isomorphism classes of rank n projective modules over the global sections of X .

5. HOMOTOPY SHEAVES

The number one tool for unstable motivic homotopy theory are the homotopy sheaves.

Definition 5.0.1. Let $X \in \text{Sh}^{\text{Nis}}(\text{Sm}_S)$. Then $\pi_0(X) \in \text{Sh}^{\text{Nis}}(\text{Sm}_S; \text{Set})$ is defined as the Nisnevich sheafification of the presheaf $U \mapsto \pi_0(X(U))$.

If $(X, x) \in \text{Sh}^{\text{Nis}}(\text{Sm}_S)_*$ is a pointed Nisnevich sheaf, then $\pi_n(X, x) \in \text{Sh}^{\text{Nis}}(\text{Sm}_S; \text{Grp})$ is defined as the Nisnevich sheafification of the presheaf $U \mapsto \pi_n(X(U), x(U))$.

Remark 5.0.2. The homotopy sheaves arise from the Postnikov tower of the topos $\text{Sh}^{\text{Nis}}(\text{Sm}_S)$ in the sense of [Lur09, 6.5.1]. If S has finite Krull dimension then $\text{Sh}^{\text{Nis}}(\text{Sm}_S)$ has finite homotopical dimension, so the homotopy sheaves jointly detect equivalences. Moreover, Postnikov towers converge.

To simplify the theory we assume that $S = \mathrm{Spec}(k)$ is the spectrum of a field. Let us cite are hard theorem due to Morel:

Theorem 5.0.3. *If $X \in \mathrm{Spc}(k)_*$ then $\pi_1(X, x)$ is strongly $\mathbb{A}^1$ -invariant.*

By the following corollary we can set up obstruction theory (k -invariants, ...) in $\mathrm{Spc}(k)_*$ just as we are used to in Spc :

Corollary 5.0.4. *Let k be a perfect field.*

- 1) *If $X \in \mathrm{Spc}(k)_*$ then $\pi_n(X, x)$ is strictly $\mathbb{A}^1$ -invariant for all $n \geq 2$.*
- 2) *Let $X \in \mathrm{Spc}(k)_*$ be connected. Then for every $n \geq 1$ the sheaf $X_{\leq n}$ is $\mathbb{A}^1$ -invariant.*

Proof. (1): By [Theorem 5.0.3](#) the sheaf $\pi_n(X) \cong \pi_1(\Omega^{n-1}X)$ is a strongly $\mathbb{A}^1$ -invariant sheaf of abelian groups. By [Theorem 4.0.1](#) it is strictly $\mathbb{A}^1$ -invariant. (2): Inductively apply [Lemma 5.0.5](#) below to the fiber sequence $K(\pi_{n+1}(X), n+1) \rightarrow X_{\leq n+1} \rightarrow X_{\leq n}$. $\square$

Lemma 5.0.5. Let $F \rightarrow E \rightarrow B$ be a fiber sequence in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_k)_*$ with B connected and $\mathbb{A}^1$ -invariant. Then F is $\mathbb{A}^1$ -invariant if and only if E is $\mathbb{A}^1$ -invariant.

Proof. We have a morphism of fiber sequences with c an equivalence:

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow a & & \downarrow b & & \downarrow c \\ F(- \times \mathbb{A}^1) & \longrightarrow & E(- \times \mathbb{A}^1) & \longrightarrow & B(- \times \mathbb{A}^1) \end{array}$$

Since B is connected a is an equivalence if and only if b is an equivalence [[Lur09](#), Lemma 6.2.3.16]. $\square$

Here is another fact which I want to leave here for general culture:

Theorem 5.0.6 (Morel, Unstable Connectivity Theorem). *If $X \in \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_k)$ is n -connected then so is $L_{\mathrm{Mot}}(X)$.*

6. HOMOTOPY T-STRUCTURE

The homotopy t-structure is a t-structure on $\mathcal{SH}(S)$ over a general base scheme. The homotopy t-structure is not the only t-structure on versions of $\mathcal{SH}(S)$ that is used in practice (e.g. Chow t-structure). The homotopy t-structure is good for “standard connectivity business” like convergence of the Adams Novikov spectral sequence for bounded below objects. Morally, the homotopy t-structure is the t-structure on $\mathcal{SH}(S)$ compatible with homotopy sheaves. By analogy the homotopy t-structure should behave like the standard t-structure on spectra. In fact, the real Betti realization $\mathrm{Be}_{\mathbb{R}} : \mathcal{SH}(\mathbb{R}) \rightarrow \mathrm{Sp}$ is t-exact.

Let $\mathcal{SH}(S)_{\geq 0}$ denote the full subcategory of $\mathcal{SH}(S)$ generated under small colimits and extensions by

$$\{\Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} X : X \in \mathrm{Sm}_S \text{ and } k \in \mathbb{Z}\}.$$

By [[Lur12](#), 1.4.4.11] $\mathcal{SH}(S)_{\geq 0}$ is the connective part of a unique t-structure on $\mathcal{SH}(S)$, called the *homotopy t-structure*.

Lemma 6.0.1. Let $f : T \rightarrow S$ be a smooth morphism, then $f^* : \mathcal{SH}(S) \rightarrow \mathcal{SH}(T)$ is t-exact.

Proof. We need to prove that $f^*(\mathcal{SH}(S)_{\geq 0}) \subseteq \mathcal{SH}(T)_{\geq 0}$ and $f^*(\mathcal{SH}(S)_{\leq 0}) \subseteq \mathrm{Sh}(T)_{\leq 0}$. The first inclusion follows because for all $X \in \mathrm{Sm}_S$ we have

$$f^*(\Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} X) \simeq \Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} (X \times_S T).$$

By adjunction, the inclusion $f^*(\mathcal{SH}(S)_{\leq 0}) \subseteq \mathrm{Sh}(T)_{\leq 0}$ is equivalent to the inclusion $f_{\#}(\mathcal{SH}(T)_{\geq 0}) \subseteq \mathcal{SH}(S)_{\geq 0}$. But, for any $X \in \mathrm{Sm}_T$ and $k \in \mathbb{Z}$ we have an equivalence

$$f_{\#}(\Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} X) \simeq \Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} f_{\#}(X) = \Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} X$$

in $\mathcal{SH}(S)$. $\square$

The following theorem explains the name of the homotopy t-structure:

Theorem 6.0.2 (Morel's Connectivity Theorem). *Let k be a field and $E \in \mathcal{SH}(k)$. Then*

(1) *$E \in \mathcal{SH}(k)_{\geq d}$ if and only if the homotopy sheaf*

$$\pi_{p,q}(E) := \pi_0(\Omega^{\infty} \Sigma^{-p,-q} E)$$

is zero for all $p - q < d$.

(2) *$E \in \mathcal{SH}(k)_{\leq d}$ if and only if $\pi_{p,q}(E) = 0$ for all $p - q > d$.*

Proof Sketch. Consider the class of spectra E such that $\pi_{p,q}(E) = 0$ for $p - q > 0$. This class is closed under colimits and extensions, and thus defines the connective part of a t-structure on $\mathcal{SH}(k)$, which we denote by $\mathcal{T}$. The non-connective part of this t-structure is given by the right-hand side of (2) (we black-box this fact).

It then suffices to prove the implications from left to right in (1) and (2).

For (1), take any $F \in \mathcal{SH}(k)_{\geq 0}$. By closure under extensions and colimits, we may assume $F = \Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} X$ for some $X \in \mathrm{Spc}(k)$. We must show that $[F, E] = 0$ for every $\mathcal{T}$ -negative E . Finally, being $\mathcal{T}$ -negative is equivalent to having $\Omega^{\infty} \Sigma_{\mathbb{G}_m}^k(E) \simeq *$ for all $k \in \mathbb{Z}$, by conservativity of homotopy sheaves.

For (2), let $E \in \mathcal{SH}(k)_{\leq -1}$, i.e., $[F, E] = 0$ for all $F \in \mathcal{SH}(k)_{\geq 0}$. Letting F run through all $\Sigma_{\mathbb{G}_m}^k \Sigma_+^{\infty} X$ shows that $\Omega^{\infty} \Sigma_{\mathbb{G}_m}^k(E) \simeq *$, so E is $\mathcal{T}$ -negative. $\square$

7. APPENDIX: BETTI REALIZATION

Construction 7.0.1 (Sketch). Equipping the $\mathbb{C}$ -points of a finite type smooth $\mathbb{R}$ -scheme with the analytic topology defines a functor of ordinary categories

$$\mathrm{Sm}_{\mathbb{R}} \rightarrow \mathrm{Top}^{C_2}, \quad X \mapsto X(\mathbb{C})^{\mathrm{an}}$$

where the C_2 -action comes from complex conjugation. Here Top^{C_2} denotes a nice enough full subcategory of the ordinary category of topological spaces equipped with a C_2 -action and C_2 -equivariant maps (e.g. the category of C_2 -CW-complexes). Let $\mathrm{Top}^{C_2} \rightarrow \mathrm{Spc}_{C_2}$ be the Dwyer-Kan localization to genuine C_2 -spaces, i.e. the localizations at maps which induce homotopy equivalences on C_2 -fixed points, as well as, on underlying spaces. Then the composite

$$\mathrm{Sm}_{\mathbb{R}} \rightarrow \mathrm{Top}^{C_2} \rightarrow \mathrm{Spc}_{C_2}$$

satisfies Nisnevich descent. Indeed, one checks that Nisnevich squares are sent to pushouts squares of topological spaces with a cofibrant leg in the genuine model structure on Top^{C_2} . We obtain a unique colimit preserving functor $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_{\mathbb{R}}) \rightarrow \mathrm{Spc}_{C_2}$ extending the previous construction. The latter functor also preserves products as $\mathrm{Sm}_{\mathbb{R}} \rightarrow \mathrm{Spc}_{C_2}$ and L_{Nis} do. Note that $\mathbb{A}^1 \in \mathrm{Sm}_{\mathbb{R}}$ is sent to $\mathbb{C} \in \mathrm{Spc}_{C_2}$ equipped with the conjugation action. This genuine C_2 -space is C_2 -equivariantly contractible.

Definition 7.0.2. Hence, we can define the *Betti realization* as the unique colimit preserving functor

$$\mathrm{Be} : \mathrm{Spc}(\mathbb{R}) \rightarrow \mathrm{Spc}_{C_2}$$

extending $\mathrm{Sm}_{\mathbb{R}} \rightarrow \mathrm{Spc}_{C_2}$, $X \mapsto X(\mathbb{C})^{\mathrm{an}}$ along the Yoneda embedding. The composite functor

$$\mathrm{Be}_{\mathbb{R}} : \mathrm{Spc}(\mathbb{R}) \rightarrow \mathrm{Spc}_{C_2} \xrightarrow{(-)^{C_2}} \mathrm{Spc}$$

is called *real Betti realization*. The composite functor

$$\mathrm{Be}_{\mathbb{C}} : \mathcal{Spc}(\mathbb{R}) \rightarrow \mathcal{Spc}_{C_2} \xrightarrow{\mathrm{Res}_1^{C_2}} \mathcal{Spc}$$

is called *complex Betti realization*.

Notation 7.0.3. We don't change notation for the induced continuous symmetric monoidal functors $\mathcal{Spc}(\mathbb{R})_* \rightarrow \mathcal{Spc}_{(C_2),*}$

Example 7.0.4. For any $X \in \mathrm{Sm}_{\mathbb{R}}$ we have $\mathrm{Be}_{\mathbb{R}}(X) = X(\mathbb{R})^{\mathrm{an}}$ the space of $\mathbb{R}$ -points equipped with the euclidean topology. For example $\mathrm{Be}_{\mathbb{R}}(\mathbb{G}_m) \simeq S^0$ and $\mathrm{Be}_{\mathbb{C}}(\mathbb{G}_m) \simeq S^1$. More generally, $\mathrm{Be}_{\mathbb{R}}(S^{d,p}) \simeq S^{d-p}$ while $\mathrm{Be}_{\mathbb{C}}(S^{d,p}) \simeq S^d$.

Because the functors $L_{\mathbb{A}^1}$ and $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Sm}_{\mathbb{R}}) \rightarrow \mathcal{Spc}_{C_2}$ preserves products, Betti realization preserves products as well.

Lemma 7.0.5. The C_2 -space $\mathrm{Be}(\mathbb{G}_m) \simeq S^{\tau}$ is equivalent to the representation sphere of the tautological orthogonal C_2 -representation τ .

We conclude that the symmetric monoidal composite functor

$$\mathcal{Spc}(\mathbb{R})_* \xrightarrow{\mathrm{Be}} \mathcal{Spc}_{C_2,*} \xrightarrow{\Sigma^{\infty}} \mathrm{Sp}_{C_2} := \mathcal{Spc}_{C_2,*}[(S^1)^{\wedge -1}, (S^{\tau})^{\wedge -1}]$$

factors through a unique continuous symmetric monoidal functor

$$\mathrm{Be} : \mathcal{SH}(\mathbb{R}) \rightarrow \mathrm{Sp}_{C_2} \quad \text{with} \quad \mathrm{Be} \circ \Sigma_+^{\infty} \simeq \Sigma_+^{\infty} \circ \mathrm{Be}.$$

Definition 7.0.6. We define *real Betti realization* as the composite

$$\mathrm{Be}_{\mathbb{R}} : \mathcal{SH}(\mathbb{R}) \rightarrow \mathrm{Sp}_{C_2} \xrightarrow{\Phi^{C_2}} \mathrm{Sp}$$

of Betti realization with geometric fixed points. The composite functor

$$\mathrm{Be}_{\mathbb{C}} : \mathcal{SH}(\mathbb{R}) \rightarrow \mathrm{Sp}_{C_2} \xrightarrow{\mathrm{Res}_1^{C_2}} \mathrm{Sp}$$

is called *complex Betti realization*.

Let ρ denote the morphism $\{\pm 1\} \rightarrow \mathbb{G}_m$ in $\mathcal{Spc}(\mathbb{R})_*$. Then, ρ becomes an equivalence after real Betti realization. Tom Bachmann proved that the induced functor

$$\mathrm{Be}_{\mathbb{R}} : \mathcal{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathrm{Sp}$$

is an equivalence. We conclude that for any $E \in \mathcal{SH}(R)$ and $p - q \in \mathbb{Z}$:

$$\begin{aligned} \pi_{p-q}(\mathrm{Be}_{\mathbb{R}} E) &\cong [\Sigma^{\infty} S^{p-q}, E[\rho^{-1}]]_{\mathcal{SH}(\mathbb{R})} = [\Sigma^{\infty} S^{p-q}, \mathrm{colim}_{n \geq q} \mathbb{G}_m^{\otimes n-q} \otimes E] \\ &= \mathrm{colim}_{n \geq q} \pi_{p-n, q-n}(E)(*). \end{aligned} \tag{2}$$

Proposition 7.0.7. *The functor $\mathrm{Be}_{\mathbb{R}} : \mathcal{SH}(\mathbb{R}) \rightarrow \mathrm{Sp}$ is t -exact for the homotopy t -structure on $\mathcal{SH}(\mathbb{R})$.*

Proof. This follows from Morel's connectivity theorem and the previous [Equation \(2\)](#). $\square$

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