SIX FUNCTORS FOR ÉTALE SHEAVES

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Contents

1. Introduction

Example 1.0.1. *Consider a smooth $\mathbb{R}$-manifold M. Let $f : M \to *$ be the map to the point, which participates in the 6ff of sheaves on locally compact topological spaces. It turns out that the dualizing sheaf*

$$f^!(\mathbb{S}) \quad \in \mathrm{Sh}(M;\mathrm{Sp})$$

lies in the image of the fully faithful functor $\mathrm{Fun}(M,\mathrm{Sp}) \hookrightarrow \mathrm{Sh}(M,\mathrm{Sp})$ *(this functor picks out the locally constant sheaves, see* [Cla25, Theorem 5.8.]*.) In fact, $f^!(\mathbb{S})$ identifies with the functor*

$$\mathbb{S}^{\mathrm{T}M} : M \to \mathrm{Sp}, \qquad m \mapsto \Sigma^\infty\left(\mathrm{T}_m M/(\mathrm{T}_m M \setminus 0)\right)$$

that sends a point m to the suspension spectrum of the one-point compactification of the tangent space at m. Because f is 6ff-smooth, the left adjoint $f_\#$ of f^* is given by $f_!(f^!(\mathbb{S}) \otimes (-))$. Note that $f_\#$ restricts to the colimit functor $\mathrm{Fun}(M,\mathrm{Sp}) \to \mathrm{Sp}$. *Hence, plugging in $\mathbb{S} \in \mathrm{Sh}(M;\mathrm{Sp})$ into $f_\#$ we obtain* ***Atiyah duality****:*

$$\Sigma^\infty_+(M) \simeq f_\#(\mathbb{S}) \simeq f_!(\mathbb{S}^{\mathrm{T}M}) = \Gamma_c(M;\mathbb{S}^{\mathrm{T}M}) \qquad \in \mathrm{Sp}\,.$$

If M is R-orientable, i.e. $\mathbb{S}^{\mathrm{T}M} \otimes R \simeq R[d]$, then base-changing the above equivalence to R-coefficients yields Poincare duality:

$$H_{d-*}(M;R) \simeq H^*_c(M;R).$$

Goal: Generalize the previous example to certain smooth F-analytic Artin stacks X, for a local field F. For example, if $X = \mathrm{B}G$ is the classifying stack of a p-adic Lie group G, our discussion will lead to Poincare duality for the continuous group homology of G with $\mathbb{S}_p$-coefficients.

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We start by setting up the six functor formalism.

2. The six functor formalism for étale sheaves

2.1. Construction of the 6ff.

Recollection 2.1.1. Recall that for $X \in \mathrm{CondAn}^{\mathrm{light}}$ we defined $\mathrm{Sh}_v(X) := \mathrm{CondAn}^{\mathrm{light}}_{/X}$ as slice category. Moreover, $\mathrm{Sh}_{\mathrm{et}}(X) \subseteq \mathrm{Sh}_v(X)$ is the full subcategory spanned by étale maps to X.[1]

Definition 2.1.2. Let $\mathcal{C}$ be a compactly-generated presentable category. We define

$$\mathrm{Sh}_{\mathrm{et}}(X;\mathcal{C}) := \mathrm{Sh}_{\mathrm{et}}(X) \otimes \mathcal{C} \qquad \text{and} \qquad \mathrm{Sh}_v(X;\mathcal{C}) := \mathrm{Sh}_v(X) \otimes \mathcal{C}\,.$$

[Lur12, Proposition 4.8.17] describes an isomorphism

$$\mathcal{X} \otimes \mathcal{C} \longrightarrow \mathrm{Fun}^{\mathrm{lex}}((\mathcal{C}^{\aleph_0})^{\mathrm{op}}, \mathcal{X}) \tag{1}$$

naturally those functors $\mathcal{X} \to \mathcal{Y}$ of presentable categories which preserve colimits and finite limits. Thus, we can reduce the following statements to $\mathcal{C} = \mathrm{An}$.

Lemma 2.1.3. Let $\mathcal{C}$ be a compactly-generated[2] presentable category.

(1) The functor $\mathrm{Sh}_{\mathrm{et}}(X;\mathcal{C}) \to \mathrm{Sh}_v(X;\mathcal{C})$ is fully faithful.
(2) The functors $X \mapsto \mathrm{Sh}_{\mathrm{et}}(X;\mathcal{C})$ and $X \mapsto \mathrm{Sh}_v(X,\mathcal{C})$ from $\mathrm{CondAn}^{\mathrm{light}}$ to Pr^L preserves arbitrary limits.
(3) Let $X \in \mathrm{CondAn}^{\mathrm{light}}$. A map $\mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}_{\mathrm{et}}(X;\mathcal{C})$ is an isomorphism in and only if $x^*\mathcal{F} \to x^*\mathcal{G}$ is an isomorphism for all $x : * \to X$

Proof. The case of $\mathcal{C} = \mathrm{An}$ has been proven in Kaif's talk. □

Remark 2.1.4. Part (2) implies that $X \mapsto \mathrm{Sh}_{\mathrm{et}}(X;\mathcal{C})$ is right Kan extended from

$$\mathrm{Pro}^{\aleph_0}(\mathrm{Fin})^{\mathrm{op}} \to \mathcal{C}, \qquad S \mapsto \mathrm{Sh}(S).$$

This is precisely the construction that we discussed in talk 4 and talk 5, i.e. for any $\mathbb{E}_\infty$-ring Λ we introduced the notation

$$\mathrm{Sh}_{\mathrm{et}}(-;\mathrm{Mod}_\Lambda) = D(-,\Lambda) : (\mathrm{CondAn}^{\mathrm{light}})^{\mathrm{op}} \to \mathrm{Cat}\,.$$

in [HM24].

Lemma 2.1.5. Let $f : T \to S$ be a map of light profinite sets. Then, the right adjoint $f_* : \mathrm{Sh}_{\mathrm{et}}(T,\mathrm{Sp}) \to \mathrm{Sh}_{\mathrm{et}}(S,\mathrm{Sp})$ to the pullback f^* of étale sheaves preserves all colimits, satisfies the projection formula and commutes with base-change along any map of light profinite sets.

Proof. We have seen this in the context of [HM24] via the identification $\mathrm{Sh}_{\mathrm{et}}(S,\mathrm{Sp}) = \mathrm{Mod}_{\Gamma(S;\mathrm{Sp})}$. We also have seen this in the context of the 6ff of sheaves on compact Hausdorff spaces, via the identification $\mathrm{Sh}_{\mathrm{et}}(S;\mathrm{Sp}) = \mathrm{Sh}(S,\mathrm{Sp})$. □

Remark 2.1.6. For any compactly-generated $\mathcal{R} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ if we tensor this Lemma with $\mathcal{R}$ we deduce the analogous lemma for $\mathcal{R}$-valued étale sheaves.

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[1] Recall that $f : X \to Y$ is called étale if for any profinite set S the pullback of f along any map $g : S \to X$ lies in the image of the fully faithful functor $\delta_S : \mathrm{Sh}(S) \to \mathrm{Sh}_v(S)$.

[2] By passing to retracts in Pr^L we can generalize this Lemma to compactly-assembled presentable coefficient categories $\mathcal{C}$, e.g. sheaves on the real line.

Construction 2.1.7 (The 6ff)**.** We obtain a unique six functor formalism on $\mathrm{Pro}^{\aleph_0}(\mathrm{Fin})$ such that the underlying functor is $\mathrm{Sh}_{\mathrm{et}}(-,\mathcal{R})$ and every map is proper (in particular, all maps f are !-able and $f_! = f_*$.) We apply [HM24, Theorem 3.4.11.] to construct a favorable class of morphisms E in $\mathrm{CondAn}^{\mathrm{light}}$ together with a sheafy extension

$$M : \mathrm{Span}_E(\mathrm{CondAn}^{\mathrm{light}}) \longrightarrow \mathrm{Pr}^L$$

of our six functor formalism to light condensed anima.

In particular, the underlying functor of our six functor formalism identifies with $\mathrm{Sh}_{\mathrm{et}}(-,\mathcal{R})$.

Definition 2.1.8. We call morphisms in E $\mathcal{R}$-!-able. We call a morphism f of condensed anima $\mathcal{R}$-suave ($\mathcal{R}$-smooth, $\mathcal{R}$-étale, $\mathcal{R}$-prim, $\mathcal{R}$-proper) if it is suave (smooth, étale, prim, proper) for this 6ff.

Remark 2.1.9. The class of maps E is produced by starting with the class E_0 of maps of light condensed anima whose every pullback to a light profinite set is light profinite, and then iteratively extending E_0 by two simple extension procedures (and, finally, taking the union over the chain of all previously constructed classes of maps). The extension procedures are as follows:

(1) Given a class of maps E', enlarge to $E'_!$, the class of maps which are !-locally on the source in E'.
(2) Given a class of maps E', enlarge it to E'_*, the class of maps such that each pullback to a light profinite set lies in E'.

Remark 2.1.10 (Change of Coefficients)**.** Suppose that $\mathcal{R} \to \mathcal{R}'$ is a morphism in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, where both $\mathcal{R}$ and $\mathcal{R}'$ are compactly generated. Then, an $\mathcal{R}$-!-able morphism is $\mathcal{R}'$-!-able. This follows by induction on Remark 2.1.9 using in each step that a $\mathcal{R}$-!-cover is also a $\mathcal{R}'$-!-cover because $- \otimes_{\mathcal{R}} \mathcal{R}'$ preserves colimits.
We conclude that if we restrict the $\mathcal{R}'$-valued 6ff to the $\mathcal{R}$-!-able maps we obtain the coefficient-change $M \otimes_{\mathcal{R}} \mathcal{R}'$ of the $\mathcal{R}$-valued 6ff M. In particular, if f is $\mathcal{R}$-smooth ($\mathcal{R}$-etale, ...), then it is $\mathcal{R}'$-smooth ($\mathcal{R}'$-étale,...).

Remark 2.1.11. Let X be a topological space, such that each point admits a secound-countable compact Hausdorff neighborhood. Then, $\mathrm{Sh}_{\mathrm{et}}(X)$ is the Postnikov completion of $\mathrm{Sh}(X)$, see [Cla25, Lemma 4.20]. If X is of finite covering dimension, then $\mathrm{Sh}(X)$ is already Postnikov complete. In particular, $\mathrm{Sh}_{\mathrm{et}}(M) = \mathrm{Sh}(M)$ for any F-manifold M and locally compact field F. Finally, Adam's uniqueness theorem for 6ffs implies that the etale sheaf 6ff $\mathrm{Sh}_{\mathrm{et}}(X;\mathcal{R})$ restricted to such spaces X agrees with the Postnikov-completed sheaf 6ff $\widehat{\mathrm{Sh}}(X;\mathcal{R})$.

In what follows, we will be interested in $\mathrm{Sh}_{\mathrm{et}}(BG;\mathcal{R})$, so we will have to leave the realm of topological spaces.

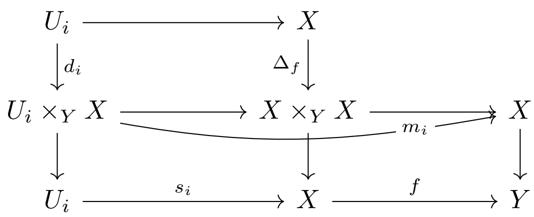
2.2. Examples of étale maps.

Proposition 2.2.1. *Any truncated etale map $f : X \to Y$ of light condensed anima is* Sp*-etale. In particular, this holds for local homeomorphisms (e.g. open inclusions) of topological spaces.*

Lemma 2.2.2. For every light profinite set S and clopen subset $f : U \subseteq S$, the inclusion $U \to S$ is Sp-étale.

Proof Sketch. When U is clopen, then f is proper. Moreover, $\mathrm{Sh}(S,\mathrm{Sp}) = \mathrm{Sh}(S-U,\mathrm{Sp}) \times \mathrm{Sh}(U,\mathrm{Sp})$ and f^* is the projection. Thus, the left and right adjoint of f^* identify with $0 \times \mathrm{id}$. So, f^* is the *right* adjoint of $f_* = f_!$. Hence, $f^! = f^*$ and one can check that this identification is induced by correct natural transformation. Note that Δ_f is an isomorphism. □

Proof of Proposition 2.2.1. A map is Sp-étale if its Sp-étale after pullback to an arbitrary light profinite set. Thus, we can assume that Y is a light profinite set. We proceed by induction on the truncation of n. Suppose we know the result for n and f is $n+1$-truncated. Because $f : X \to Y$ is etale we can find a collection of clopen subsets $U_i \subseteq Y$ and local sections $s_i : U_i \to X$ of f, which are jointly surjective.
Indeed, we have an explicit identification $f \in \mathrm{et}_{/Y} = \mathrm{Sh}_{\mathrm{et}}(Y) = \mathrm{Sh}(Y)$, so that $f = \mathrm{colim}_i U_i$ for some clopen sets U_i of Y, as clopen sets generate $\mathrm{Sh}(Y)$ under colimits (they are a basis for the topology on Y). The map $U_i \to f$ in $\mathrm{et}_{/Y}$ yields the section $s_i : U_i \to Y$ of f. The colimit formula translates to the s_i being jointly surjective. The following diagram



consists of pullback squares. Because Δ_f is n-truncated and étale, it is Sp-etale by induction. The inclusion $U_i \hookrightarrow Y$ is Sp-etale by Lemma 2.2.2. Hence, also $s_i = m_i \circ d_i : U_i \to X$ is is Sp-etale. We conclude that $\{s_i : U_i \to X\}$ is a jointly surjective Sp-étale cover and therefore detects Sp-etaleness. But the composite $U_i \to Y$ is Sp-etale by Lemma 2.2.2. □

2.3. **Examples of proper maps.**
Recall the following criterion for properness:

Lemma 2.3.1 ([HM24, Cor. 4.7.5])**.** Let $\mathcal{R}$ serve as coefficients for our 6ff on $\mathrm{CondAn}^{\mathrm{light}}$. Let $g : X' \to X$ be an $\mathcal{R}$-proper map. Suppose $g_* \mathbb{1} \in \mathrm{CAlg}(\mathrm{Sh}_{\mathrm{et}}(X;\mathcal{R}))$ is descendable and $f : X \to Y$ is truncated. Suppose $f \circ g$ is $\mathcal{R}$-proper, then f is proper.

Even though we already have seen a version of the following theorem (modulo Postnikov-completion) when we discussed sheaves on compact Hausdorff spaces, we are interested in its proof.

Theorem 2.3.2. *Suppose X is a light condensed anima such that either:*

1. *(1) X is a second-countable compact Hausdorff space of finite cohomological dimension (i.e. there is a $d \in \mathbb{N}$ such that for all $M \in \mathrm{Sh}_{\mathrm{et}}(X; D(\mathbb{Z}))$ concentrated in degree 0, we have $\Gamma(M) \in D(\mathbb{Z})_{\geq -d}$); or*
2. *(2) X is a 1-truncated light profinite anima of finite cohomological dimension. (light profinite anima = countable inverse limit of π-finite anima). For example, $X = \mathrm{B}G$ for a light profinite group of finite cohomological dimension and, as explained in Kaif's talk, $X = \mathrm{B}H$ the relative classifying stack of a relative uniform pro-p-group over a p-adic manifold.*

Then, the map

$$f : X \to *$$

is Sp*-proper.*

Proof. There exists a light profinite set and a surjection $g : S \twoheadrightarrow X$, such that for any $T \to X$ with $T \in \mathrm{Pro}^{\aleph_0}(\mathrm{Fin})$ the fiber product $T \times_X S$ is also a light profinite set.
(Every compact metric space is a quotient of a Cantor space, see nlab article on Cantor spaces. In case (2), g is constructed inductively along the sequential inverse limit, as in [Cla25, 4.23.])
Because any map of light profinite sets is proper, $f \circ g$ and g are proper. We need to show that $g_*(\mathbb{1}) \in \mathrm{Sh}_{\mathrm{et}}(X, \mathrm{Sp})$ is descendable.

Let's blackbox [Cla25, Theorem 6.21] **stalkwise criterion for descendability**: If $\mathrm{Sh}_{\mathrm{et}}(X, \mathrm{Sp})$ is countably assembled and rigid and X has finite cohomological dimension, then $A \in \mathrm{Sh}_{\mathrm{et}}(X, \mathrm{Sp})$ is descendable if and only if there is $N \in \mathbb{N}$ such that for all maps $x : * \to X$, the pullback x^*A is descendable of index $\leq N$.
Using finite cohomological dimension a certain composite of A-null maps will be phantom. By countable-assembly a composition of 2 phantom maps is null.
Let's believe without proof[3] that $\mathrm{Sh}_{\mathrm{et}}(X, \mathrm{Sp})$ is countably-assembled and rigid, so that the blackbox applies. Hence, the theorem follows from the following Lemma 2.3.3. □

Lemma 2.3.3. Let $g : S \to X$ be a $\mathcal{R}$-proper, surjective map of condensed anima. Then, for all $x : * \to X$ the algebra $x^*g_*(\mathbb{1}) \in \mathcal{R}$ is descendable of index ≤ 1.

Proof. By proper-base change $x^*g_*(\mathbb{1}) = (g_x)_* \mathbb{1}$ for $g_x : g^{-1}(x) \to *$. Since g is surjective, $g^{-1}(x)$ is nonempty. Any nonempty condensed anima admits a point[4], so g_x admits a section. This section induces a section of the unit map $\mathbb{S} \to (g_x)_*\mathbb{S}$, and thus $(g_x)_* \mathbb{1}$ is descendable of index ≤ 1. □

Remark 2.3.4. If $X \to *$ is $D(\mathbb{Z})$-proper, then X has finite cohomological dimension.

Corollary 2.3.5. *Let $f : X \to Y$ be a map in* $\mathrm{CondAn}^{\mathrm{light}}$. *If $\{Y_i \to Y\}_{i\in I}$ is a jointly surjective family of maps to Y and $X \times_Y Y_i \to Y_i$ is !-able for all i, then f is !-able. The same holds for the classes of $\mathcal{R}$-proper, $\mathcal{R}$-smooth, and $\mathcal{R}$-étale maps.*

Proof. The same argument as in Theorem 2.3.2 shows that for any surjective map $f : T \twoheadrightarrow S$ of light profinite sets $f_*\mathbb{S}$ is descendable in $\mathrm{Sh}_{\mathrm{et}}(S, \mathrm{Sp})$. By [HM24, 4.7.4.], f is a universal !-cover. Finite disjoint unions are also !-covers by Lemma 2.2.2. Hence, every cover of a light profinite set by light profinite sets is a !-cover.

To prove the corollary we can assume that Y is a light profinite set, as these notions can be checked after pullback to such. For each $i \in I$ we choose a jointly surjective family $\{S_{i,j} \to Y_i\}_{j\in J_i}$ with $S_{i,j}$ a light profinite set. Then, $\{S_{i,j} \to Y\}$ is a jointly surjective family of maps of light profinite sets, hence a !-cover. Thus, it suffices to check that the pullback $X \times_Y S_{i,j} \to S_{i,j}$ lies in the respective class of maps. But this map is the pullback of $X \times_Y Y_i \to Y_i$ along $S_{i,j} \to Y_i$. □

2.4. Examples of !-able maps.

Corollary 2.4.1. *Let $f : X \to Y$ be either*

a) a map between topological spaces that admit a basis by second countable Hausdorff spaces of finite cohomological dimension.

b) a representable map[5] in $\mathrm{Sh}(\mathrm{Man}_F)$ for some local field F.

Then f is Sp-*!-able.*

Proof. a): By cancellation it suffices to show that $X, Y \to *$ are !-able. Open maps are smooth and the cover is jointly conservative, so that we can assume that X satisfies the assumptions of Theorem 2.3.2, so that $X \to *$ is proper.
b): The case a) implies that any map of F-manifolds is !-able. Take a jointly surjective family $\{M_i \to Y\}$ of F-manifolds. By Corollary 2.3.5 we can test after pullback to each M_i, but this yields a map of F-manifolds. □

[3]For a second countable Hausdorff space X the category $\mathrm{Sh}_{\mathrm{et}}(X; \mathrm{Sp})$ is the left completion of $\mathrm{Sh}(X; \mathrm{Sp})$

[4]Qi's argument: Say there is no map $* \to X$. Then, $X(*) = \emptyset$ by Yoneda. Let S be light profinite. Then, there is a map $* \to S$, which induces $X(S) \to X(*) = \emptyset$. So, $X(S) = \emptyset$, as well. (This works for any category of sheaves on a site $\mathcal{C}$ where every object in $\mathcal{C}$ admits a map from the point.)

[5]For any F-manifold M the sheaf $M \times_Y X$ is represented by an F-manifold.

3. Smooth Morphisms

We blackbox the following criterion for smoothness:

Lemma 3.0.1 ([Cla25, Prop. 8.4])**.** Let $\mathcal{R} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ be semi-rigid (compactly-assembled and the right adjoint of the tensor product preserves colimits and is $\mathcal{R}$-linear.)(e.g. any rigid category or $\mathrm{Sp}_{\hat{p}}$). Suppose a map $f : X \to S$ with $S \in \mathrm{Pro}^{\aleph_0}(\mathrm{Fin})$ is $\mathcal{R}$-!-able. TFAE:

(1) f is $\mathcal{R}$-smooth;
(2) $f^!$ commutes with colimits and $f^! \mathbb{1}$ is invertible.

Furthermore, if we just assume that $f^!$ commutes with colimits, then the formation of $f^!$-commutes with base-change, so we can test invertibility of $f^! \mathbb{1}$ locally on S.

The Lemma follows from [HM24, Lemma 4.5.4] and arguments (experts would call these arguments standard) about base-change and linearity in Pr^L.

Theorem 3.0.2 (The Archimedean Case)**.** *(1) Suppose that $f : X \to Y$ is a (surjective) representable submersion in $\mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$, then f is* Sp*-smooth (and a* Sp*-!-cover).*
(2) For every $\mathbb{R}$-analytic smooth Artin stacks X, i.e. there exists a surjective representable submersion $M \to X$ with $M \in \mathrm{Man}_{\mathbb{R}}$, the map $X \to *$ is Sp*-smooth.*

Because !-able satisfy cancellation (are a geometric class), any map between $\mathbb{R}$-analytic smooth Artin stacks is Sp-!-able.

Rough Sketch. We descends to case of manifolds $X, Y \in \mathrm{Man}_{\mathbb{R}}$ via Corollary 2.3.5. On $M \in \mathrm{Man}_{\mathbb{R}}$ there is an equivalence $\mathrm{Sh}(M) \to \mathrm{Sh}_{\mathrm{et}}(M)$ of 6ffs, see Remark 2.1.11, and we already covered this when we discussed sheaves on topological spaces. Alternatively, one can descend all the way to discs and then intervals via base change. On the interval it is a direct computation using Lemma 3.0.1. □

Theorem 3.0.3 (The Non-Archimedean Case)**.** *Let X be a $\mathbb{Q}_p$-manifold, $G \to X$ a group object in submersions over X and $\mathrm{B}G \to X$ the relative classifying stack, i.e. the geometric realization in $\mathrm{Sh}(\mathrm{Man}_{\mathbb{Q}_p})$ of the action groupoid. Then:*

(i) $\mathrm{B}G \to X$ is $\mathrm{Sp}_{\hat{p}}$-smooth.
(ii) If $G \to X$ is proper, i.e. preimage of a compact set is compact, and each fiber G_x is p-torsion free, then $\mathrm{B}G \to X$ is $\mathrm{Sp}^\wedge_p$-proper.

Proof Sketch. We start with (ii). By Corollary 2.3.5 we can assume that $X = S$ is a light profinite set. By Lemma 2.3.1 applied to the diagram

$$\mathrm{id} : S \xrightarrow{e} \mathrm{B}G \to S,$$

it suffices to check

- The quotient map $e : S \to \mathrm{B}G$ (and the identity $S \to S$)is $\mathrm{Sp}_{\hat{p}}$-proper; and
- $e_* \mathbb{1} \in \mathrm{Sh}_{\mathrm{et}}(\mathrm{B}G, \mathrm{Sp}_{\hat{p}})$ is descendable.

The first point can be checked after pullback along $S \to \mathrm{B}G$, where it becomes the map $G \to S$. The latter is proper, as proper maps of p-adic manifolds are Sp-proper, by Lurie's proper-basechange (for p-adic manifolds $\mathrm{Sh}_{\mathrm{et}}(M) = \mathrm{Sh}(M)$ by Remark 2.1.11). For the second point we want to apply a p-adic version[6] of Dustin's **stalkwise criterion for descendability** [Cla25, Remark 6.22.]. We need to check:

- $e_* \mathbb{1}$ is stalkwise descendable of uniformly bounded index because e is a proper surjection, see Lemma 2.3.3.

[6] $\mathrm{Sp}_{\hat{p}}$ is not rigid, but semi-rigid.

- $\mathrm{Sh}_{\mathrm{et}}(\mathrm{B}G, \mathrm{Sp})$ is rigid and countably-assembled.
Sketch: Kaif explained that there is an etale surjection $\mathrm{B}H \to \mathrm{B}G$, where $H \to S$ is a fiber-wise uniform p-group. This reduces the statement to uniform p-groups, where we can employ their structure theorem.
- $\mathrm{B}G$ has finite mod-p cohomological dimension.
By a general argument [Cla25, Remark 6.23], it suffices to check that the fibers of $\mathrm{B}G \to S$ have uniformly bounded mod-p cohomological dimension. But the fibers are precisely the G_s, which are compact p torsion-free p-adic Lie groups, by assumption. Serre proved that these have finite mod-p cohomological dimension bounded by the dimension of G_s. This gives a uniform bound as S is compact and $\dim(G_s) : S \to \mathbb{N}$ is continuous.

In our discussion of (i) we proceed similarly. Again, we can replace X by a light profinite set S. Kaif explained ([Cla25, Proposition 5.15., 5.16.]) that there is an etale surjection $\mathrm{B}H \to \mathrm{B}G$, where $H \to S$ is a proper map and a fiber-wise uniform p-group. We can reduce to $H = G$ (!-ability and smoothness can be checked on jointly conservative smooth cover of the source, see [HM24, 4.7.1, 4.5.8]). By Lemma 3.0.1, we need to show that $f : \mathrm{B}H \to S$ is !-able, $f^!$-preserves colimits and $f^! \mathbb{1}$ is invertible. We have seen in part (ii) that f is $\mathrm{Sp}_{\hat{p}}$-proper, thus !-able.

- $f^!$ preserves colimits:
Because $\mathrm{Sh}_{\mathrm{et}}(\mathrm{B}H, \mathrm{Sp})$ is compactly generated (the structure theory of uniform pro-p groups provides explicit compact generators), it suffices to check that $f_! = f_*$ preserves compact objects. We claim that $f_! \mathbb{S}/p \in \mathrm{Sh}_{\mathrm{et}}(S, \mathrm{Sp}_{\hat{p}})$ is compact and it turns out that this is sufficient as the argument for the other explicit generators is the same. By a classical theorem on uniform pro-p groups the morphism
$$\Lambda^i_{\mathbb{F}_p} H^1 f_* \mathbb{F}_p \longrightarrow H^i f_* \mathbb{F}_p$$
is an isomorphism and the sheaf $H^1 f_* \mathbb{F}_p$ is isomorphic to $(\mathbb{F}_p)^{\dim(H)}$ (Here $\dim(H) : S \to \mathbb{N}$ is locally constant). Consequently, $f_*(\mathbb{F}_p) = f_*(\mathbb{S}/p) \otimes \mathbb{F}_p$ is locally constant, finitely generated and free. By finite cohomological dimension $f_* \mathbb{S}/p$ is bounded in the t-structure. It turns out that any p-adic etale sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(S, \mathrm{Sp}_{\hat{p}})$, which satisfies these properties and is killed by some power of p, is compact, see [Cla25, Lemma 8.8].
- $f^! \mathbb{1}$ is invertible:
Because $f^! \mathbb{S}/p$ is bounded in the t-structure (uses that $\mathrm{B}H$ is an inverse limit of p-finite anima of uniformly bounded cohomological dimension) we are left to prove that $f^! \mathbb{F}_p \in \mathrm{Sh}_{\mathrm{et}}(\mathrm{B}H, \mathbb{F}_p)$ is invertible (reduction by the general arguments in [Cla25, Lemma 8.8]). $\mathrm{Sh}_{\mathrm{et}}(\mathrm{B}H, (-) \otimes \mathbb{F}_p)$ is compactly generated by the unit (check what happens to the explicit generators above under $\otimes \mathbb{F}_p$), so
$$\mathrm{Mod}_{f_* \mathbb{F}_p}(\mathrm{Sh}_{\mathrm{et}}(S, D(\mathbb{F}_p))) \longrightarrow \mathrm{Sh}_{\mathrm{et}}(\mathrm{B}H, D(\mathbb{F}_p))$$
is an isomorphism, by the Schwede-Shipley theorem. Under this equivalence we can identify the double right adjoint:
$$f^! = \underline{\mathrm{Hom}}_{\mathbb{F}_p}(f_* \mathbb{F}_p, -).$$
We need to show $\underline{\mathrm{Hom}}_{\mathbb{F}_p}(f_* \mathbb{F}_p, \mathbb{F}_p)$ is invertible as $f_* \mathbb{F}_p$-module. But we calculated the homotopy of $f_* \mathbb{F}_p$ as the exterior algebra on a locally free sheaf of finite rank $\dim(H)$ in degree -1. Thus the claim follows from the self-duality (up to shift) of exterior algebras.

□

4. On nonstandard paths

Let F be a local field (think: $F = \mathbb{R}, \mathbb{C}$ or $F = \mathbb{Q}_p$.) We have seen that certain maps of F-analytic smooth Artin stacks $f : X \to Y$ are smooth (e.g. $\mathrm{B}G \to M$ for relative p-adic

Lie groups; or any map $X \to Y$ when $F = \mathbb{R}$). We would like to identify $f^!(\mathbb{1})$. As in the example discussed in the introduction this would produces an explicit Poincare duality formula for the (relative) cohomology of X. Recall that in our $\mathbb{R}$-manifold example the dualizing sheaf $f^!(\mathbb{1}) \in \mathrm{Pic}(\mathrm{Sh}_{\mathrm{et}}(X, \mathrm{Sp}_{\hat{p}}))$ identified with the Thomification of the relative tangent vector bundle. We hope to generalize this fact to any smooth map $f : X \to Y$ of F-analytic smooth Artin stacks. The strategy to relate the dualizing sheaf with tangent information is as follows:

The deformation to the tangent bundle, discussed in Daniel's talks, yields a $F^\times$-equivariant bundle over $F \times X$, hence, a sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X \times (F/F^\times), \mathrm{Sp})$. Its value at 0 roughly gives the tangent bundle and its value at 1 allows us to compute the dualizing sheaf. So, we need the following theorem:

Theorem 4.0.1. *Let F be a local field and p a prime. Suppose given $X \in \mathrm{CondAn}^{\mathrm{light}}$ and let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X \times (F/F^\times); \mathrm{Sp}_{\hat{p}})$ be a tensor invertible sheaf.*
Then, there is a natural isomorphism $0^* \mathcal{F} \simeq 1^* \mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X; \mathrm{Sp}_{\hat{p}})$.

Proof. We choose a map of condensed anima $\gamma : I \to F/F^\times$ with lifts $0, 1 : * \to I$ of $0, 1 \in F$ ("a path from 0 to 1") such that

$$\mathrm{pr}^* : \mathrm{Sh}_{\mathrm{et}}(X, \mathrm{Sp}_{\hat{p}}) \to \mathrm{Sh}_{\mathrm{et}}(I \times X, \mathrm{Sp}_{\hat{p}})$$

is fully-faithful and any invertible sheaf lies in the essential image of pr^*. Then any invertible sheaf $\mathcal{F}$ is of the form $(\mathrm{pr}^*)^* \mathcal{F}'$ so that

$$0^* \mathcal{F} = 0^* \mathrm{pr}^* \mathcal{F}' = \mathcal{F}' = 1^* \mathrm{pr}^* \mathcal{F}' = 1^* \mathcal{F}.$$

The rest of the talk will be devoted to constructing such a path γ. □

Remark 4.0.2. When $F = \mathbb{R}$ or $\mathbb{C}$, for any invertible $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X \times F; \mathrm{Sp})$ there is a natural isomorphism $0^* \mathcal{F} \simeq 1^* \mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X; \mathrm{Sp})$. Indeed, $\mathrm{Sh}_{\mathrm{et}}(X; \mathrm{Sp}) \to \mathrm{Sh}_{\mathrm{et}}(X \times F; \mathrm{Sp})$ is fully-faithful and any invertible sheaf lies in the image. So the argument above applies.

4.1. Exotic Intervals.

To start we discuss exotic intervals.

Definition 4.1.1. Let $I \in \mathrm{CondAn}^{\mathrm{light}}$ and $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ compactly-generated. We say that I is a $\mathcal{C}$-interval if $f : I \to *$ is $\mathcal{C}$-proper, $f^* : \mathcal{C} \to \mathrm{Sh}_{\mathrm{et}}(I, \mathcal{C})$ is fully-faithful and any invertible sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(I, \mathcal{C})$ lies in the image of f^*.

[Cla25] doesn't talk about $\mathcal{C}$-intervals but a more general thing, he calls right $\mathcal{C}$-blob. For our purposes our ad-hoc definition of $\mathcal{C}$-intervals is sufficient.

Lemma 4.1.2. Let I be a $\mathcal{C}$-interval and let $X \in \mathrm{CondAn}^{\mathrm{light}}$. Then,

$$\mathrm{pr}^* : \mathrm{Sh}_{\mathrm{et}}(X; \mathcal{C}) \to \mathrm{Sh}_{\mathrm{et}}(X \times I, \mathcal{C})$$

is fully-faithful. Moreover, $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X \times I, \mathcal{C})$ is the in the essential image of pr^* if and only if for all $x : * \to X$ the pullback of $\mathcal{F}$ along $x \times I : I \to X$ is constant, i.e. pulled back from $I \to *$. Furthermore, any invertible sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{et}}(X \times I, \mathcal{C})$ lies in the image of pr^*.

Proof. The first two statements are asking whether the unit (respectively, the counit) for $\mathrm{pr}_* \dashv \mathrm{pr}^*$ are isomorphisms. By Lemma 2.1.3 we can check on points $x : * \to X$. Let $f : I \to *$. By proper base change

$$x^*(1 \longrightarrow \mathrm{pr}_* \mathrm{pr}^*) \simeq (x^* \longrightarrow f_*(x \times I)^* \mathrm{pr}^* \simeq f_* f^* x^*) \simeq (1 \longrightarrow f_* f^*) \circ x^*.$$

and the unit $1 \to f_* f^*$ is an isomorphism by assumption. Applying an analogous argument to the counit we get the second statement. The last statement reduces to $X = *$, by the second statement. □

The Lemma implies the proof of Theorem 4.0.1 goes through for I an $\mathrm{Sp}_{\hat{p}}$-interval object. The following example gives us sufficient supply of $\mathrm{Sp}_{\hat{p}}$-intervals:

Lemma 4.1.3. Let I be a second-countable compact Hausdorff space of finite cohomological dimension such that

(1) $f : I \to *$ induces an $\mathbb{F}_p$-cohomology isomorphism and
(2) any continuous map $I \to BG$ to the classifying space of a finite group G is nullhomotopic.

Then, I is an $\mathrm{Sp}_{\hat{p}}$-interval.

Proof Sketch. By Theorem 2.3.2 f is Sp-proper. The functor f^* is fully-faithful if and only if for any $A \in \mathrm{Sp}_{\hat{p}}$, the unit $A \to f_*(f^*(A))$ is an equivalence.

Recall that any exact functor $g : \mathcal{C} \to \mathcal{D}$ between stable categories with Postnikov complete t-structure and $g(\tau_{\geq 0}\mathcal{C}) \subseteq \tau_{\geq -d}(\mathcal{D})$ preserves limits of Postnikov towers. We can apply this to $g = f_* f^*$ by our bounded cohomological dimension assumption. By properness of f the functor g preserves colimits, so that we can reduce to $A = \mathbb{F}_p$, where

$$\pi_{-*}(\mathbb{F}_p \to f_*(\mathbb{F}_p)) = H^*(I \to *, \mathbb{F}_p)$$

is an isomorphism by hypothesis.

We are left to proving that any dualizable $\mathcal{F} \in \mathrm{Sh}(I; \mathrm{Sp}_{\hat{p}})$ the counit $f^* f_* \mathcal{F} \to \mathcal{F}$ is an equivalence. By a more involved version of the previous argument one can reduce to locally constant sheaves of finite dimensional $\mathbb{F}_p$-vector spaces. Because I is connected, such sheaves are classified by a map $I \to \mathrm{BGL}_d(\mathbb{F}_p)$, and hence are constant by assumption. □

4.2. **The Required Nonstandard Path.** We need to construct an $\mathrm{Sp}_{\hat{p}}$-interval I, a map of condensed anima $\gamma : I \to F/F^\times$ and lifts $0, 1 \in F$ along γ. In the talk I drew a bunch of pictures. I was to lazy to put them into the notes.

Let us start with $K = \{n \in \mathbb{Z}\} \cup \{\infty\}$. As a topological space

$$K \cong \{1/n : n \in \mathbb{N}_{\geq 1}\} \cup \{0\} \subseteq \mathbb{R},$$

which is one of the pictures I drew.

Choose some $q \in F$ with $|q| < 1$ so that

$$K \to F, \qquad n \mapsto q^n$$

is continuous. $\mathbb{Z}$ acts on K by translating $\mathbb{Z}$ and the trivial action on ∞. The above map restricts to group homomorphism $\mathbb{Z} \to F^\times$ and $K \to F$ is $\mathbb{Z}$-equivariant. We get induced maps

$$K/\mathbb{Z} \to F/\mathbb{Z} \to F/F^\times$$

of quotients (these quotients are computed as realization of action groupoid in condensed anima).

Now we enlarges $K/\mathbb{Z}$ to make it an interval object (its not representable by a topological space yet, so our interval recognition criterion Lemma 4.1.3 doesn't apply.) We need to make the $\mathbb{Z}$ action free and properly discontinuous, to make the quotient quasi-seperated. Our first try is $X := (K \times \mathbb{R})/\mathbb{Z}$ where $\mathbb{Z}$ acts diagonally and by translation on $\mathbb{R}$. Then, the action is free and properly discontinuous, so that X is a space (compare [HM24, 5.3.8.]). Moreover, the obvious projection $X \to K/\mathbb{Z}$ is a surjective map of condensed anima.

To see what X looks like consider

$$K \times \mathbb{R} \cong \cup_{n \in \mathbb{N}_{\geq 1}} (\{1/n\} \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}) \subseteq \mathbb{R} \times \mathbb{R}$$

Lets analyze what the $\mathbb{Z}$ action does to that subspace of $\mathbb{R}^2$. The $\mathbb{Z}$ action doesn't leave $\{0\} \times \mathbb{R}$, so we get a circle $\{0\} \times \mathbb{R}/\mathbb{Z}$. All the other lines are identified to a helix growing out of that circle. The helix speeds up more and more the closer it comes to the boundary circle $\{0\} \times \mathbb{R}$. X admits a set decomposition into the closed subset $\{0\} \times \mathbb{R}/\mathbb{Z}$ (the circle) and an open subset homeomorphic $\mathbb{R}_{\geq 0}$ (the helix). The helix speeding up infinitely fast should remind you of the topologists sine curve.

By cutting the helix off at $(0,1] \subseteq \mathbb{R}_{>0}$ we get a compact Hausdorff space $I \subseteq X$, so that $I \to F/F^\times$ still lifts 0 and 1.

As I still contains a circle, I has two much $\mathbb{F}_p$-cohomology for our interval recognition criterion Lemma 4.1.3. In a final step we glue in a solenoid

$$S := \lim_n (S^1 \xleftarrow{(-)^2} S^1 \xleftarrow{(-)^3} S^1 \xleftarrow{(-)^4} S^1 \leftarrow \cdots)$$

along the projection $S \to S^1$ onto the first term in the inverse limit. We get a surjective projection $\tilde{I} \twoheadrightarrow I$. (Formally, we taking inverse limit of copies of X along multiplications maps and then cut the helix off to make it compact.) Now $\tilde{I}$ has decomposition into a "boundary" closed subspace homeomorphic to the solenoid and a open subset homeomorphic to $(0,1]$.

We claim that $\tilde{I}$ is an $\mathrm{Sp}_{\hat{p}}$-interval object. Indeed, is a connected second countable compact Hausdorff space (it is a countable inverse limit of such). Finally, for G a finite group any map $S \to \mathrm{B}G$ is null, because any map $S^1 \to \mathrm{B}G$ corresponds to an element $* \to \Omega\,\mathrm{B}G = G$ of G.[7] As this element is of finite order, it dies after passing along the inverse limit of the solenoid to a high enough stage.

This convinced ourselves that the assumptions[8] of Lemma 4.1.3 are satisfied for $\tilde{I}$. Consequently, $\tilde{I}$ is a $\mathrm{Sp}_{\hat{p}}$-interval object. The composite

$$\gamma : \tilde{I} \twoheadrightarrow I \to X \to K/\mathbb{Z} \to F/F^\times$$

is our required nonstandard path, which makes the proof of Theorem 4.0.1 work.

References

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[7] Here we used the fact that any G-principle bundle $P \to \lim_{\mathbb{N}} S^1$ is pulled back from some S^1, by compactness of P.

[8] I left out some details, like finite cohomological dimension