VERSIONS OF QUILLEN'S THEOREM A

FABIO NEUGEBAUER

In this essay, we use the language of ∞-categories. For us a category is an $(\infty, 1)$-category. The category of ∞-groupoids is denoted $\mathcal{S}$.

1. Introduction

Scenario 1.1. In practice, we are often confronted with the following situation: We are given a functor $F : J \to \mathcal{C}$ and we desperately want to compute $\operatorname{colim}_J F$. A plausible strategy is choosing another functor $f : I \to J$ such that $\operatorname{colim}_I(Ff)$ is computable. Then, we establish that the canonical map $\operatorname{colim}_I(Ff) \to \operatorname{colim}_J F$ is an equivalence.

To prove the latter, I "always" employ the following theorem of Quillen's

Theorem (Quillen's Theorem A)**.** *Let $f : I \to J$ be a functor between small categories. The following are equivalent:*

- *For **any** cocomplete category $\mathcal{C}$ and any functor $F : J \to \mathcal{C}$ the canonical map $\operatorname{colim}_I(Ff) \to \operatorname{colim}_J f$ is an equivalence.*
- *For all $j \in J$ the slice $I_{j/} := I \times_J J_{j/}$ has contractible realization: $|I_{j/}| \simeq *$.*

I was today's years old, when I learned that employing Quillen Theorem A is often overkill in Scenario 1.1. In many situations there is the following improvement:

Improvement 1.2 (R-linear Quillen's Theorem A)**.** *Let $f : I \to J$ be a functor between small categories and R an $\mathbb{E}_1$-ring. The following are equivalent:*

a) For all presentably R-linear categories $\mathcal{C}$ and all functors $F : J \to C$ the canonical map $\operatorname{colim}_I(F \circ f) \to \operatorname{colim}_J F$ is an equivalence.
b) For all $j \in J$ the map $|I_{j/}| \to *$ induces an isomorphism on R-homology.

In fact, we'll deduce the R-linear Quillen theorem from the following specialized version of Quillen's theorem.

Theorem A (Specialized Quillen's Theorem A, Theorem 3.5)**.** *Let $f : I \to J$ be a functor between small categories and $\mathcal{C}$ a presentable category. Choose any small set $\{c_k\}_{k \in K}$ of objects of $\mathcal{C}$, such that the functors $\{\operatorname{Map}_{\mathcal{C}}(c_k, -)\}_{k \in K}$ jointly detect equivalences. Then, the following are equivalent:*

a) For any functor $F : J \to \mathcal{C}$ the canonical map $\operatorname{colim}_I F \circ f \to \operatorname{colim}_J F$ is an equivalence in $\mathcal{C}$.
b) For any $j \in J$ and $k \in K$ the canonical[1] map

$$|I_{j/}| \otimes c_k \to c_k$$

is an equivalence in $\mathcal{C}$.

From this specialized version of Quillen's theorem we can also proof an "enriched" version.

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[1] $|I_{j/}| \otimes c_k$ is defined as the colimit of the constant functor $|I_{j/}| \to \mathcal{C}$ with value c_k.

Definition 1.3 (Presentably $\mathcal{R}$-linear categories)**.** Let $(\mathcal{R}, \otimes, \mathbb{1}_{\mathcal{R}})$ be a presentably monoidal category. We say that a presentable category $\mathcal{C}$ is *presentably $\mathcal{R}$-linear* if for any $c \in \mathcal{C}$ there is a limit preserving functor $\mathcal{R}(c,-) : \mathcal{C} \to \mathcal{R}$ together with a natural transformation making the diagram

$$\begin{tikzcd} & \mathcal{R} \arrow[dr, "\mathrm{Map}_{\mathcal{R}}(\mathbb{1}_{\mathcal{R}},-)"] & \\ \mathcal{C} \arrow[ur, dotted, "\mathcal{R}(c,-)"] \arrow[rr, "\mathrm{Map}_{\mathcal{C}}(c,-)"'] & & \mathcal{S} \end{tikzcd}$$

commute.

Example 1.4. The category $\mathcal{R}$ is presentably $\mathcal{R}$-linear, because we can take $\mathcal{R}(c,-)$ to be the right adjoint of $(-) \otimes_{\mathcal{R}} c : \mathcal{R} \to \mathcal{R}$.

Example 1.5. An $\mathcal{R}$-enriched presentable category $\mathcal{C}$ is presentably $\mathcal{R}$-linear if the enrichment $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{R}$ preserves limits in the second variable.

Theorem B ($\mathcal{R}$-linear Quillen's Theorem A, Theorem 4.1)**.** *Let $(\mathcal{R}, \otimes, \mathbb{1}_R)$ be a presentably monoidal category and $f : I \to J$ a functor between small categories. Then, the following are equivalent:*

a) For any presentably $\mathcal{R}$-linear category $\mathcal{C}$ and any functor $F : J \to \mathcal{C}$, the canonical map $\mathrm{colim}_I(F \circ f) \to \mathrm{colim}_J F$ is an equivalence.
b) For any $j \in J$, the canonical map $|I_{j/}| \otimes \mathbb{1}_R \to \mathbb{1}_R$ is an equivalence in $\mathcal{R}$.

Example 1.6. Let R be an $\mathbb{E}_1$-ring. The R-linear version 1.2 of Quillen's theorem A follows from the $\mathcal{R}$-linear version of Quillen's theorem applied to the category of left R-modules $\mathcal{R} := \mathrm{LMod}_R$.

Example 1.7. Consider the presentably monoidal category $(\mathcal{S}_{\leq n}, \times, \mathbb{1}_n)$ of n-truncated ∞-groupoids and let $f : I \to J$ be a functor of small categories. For any $j \in J$ we have

$$|I_{j/}| \otimes \mathbb{1}_n \simeq \tau_{\leq n}(|I_{j/}|)$$

in $\mathcal{S}_{\leq n}$. So, condition b) of the $\mathcal{S}_{\leq n}$-linear Quillen theorem reduces to all slices $I_{j/}$ being n-connected.

2. Generators of Presentable Categories

Proposition 2.1. *Let $\mathcal{C}$ be a presentable category and let $\{c_k\}_{k \in K}$ be a small set of objects in $\mathcal{C}$. Then the following conditions are equivalent:*

a) The corepresented functors $\{\mathrm{Map}_{\mathcal{C}}(c_k,-)\}_{k \in K}$ jointly detect equivalences in $\mathcal{C}$.
b) The category $\mathcal{C}$ itself is the smallest full subcategory of $\mathcal{C}$, which is closed under small colimits in $\mathcal{C}$ and contains the objects $\{c_k\}_{k \in K}$.

If these conditions are satisfied, we say that the set of objects $\{c_k\}_{k \in K}$ generates $\mathcal{C}$ under small colimits.

Proof. To prove *b)* $\Rightarrow$ *a)* let f be a morphism in $\mathcal{C}$ so that the induced morphism of mapping spaces $\mathrm{Map}_{\mathcal{C}}(c_k, f)$ is an equivalence for all $k \in K$. Then the full subcategory of $\mathcal{C}$ spanned by all $c \in \mathcal{C}$ for which the induced map $\mathrm{Map}_{\mathcal{C}}(c, f)$ is an equivalence is closed under colimits in $\mathcal{C}$ and contains $\{c_k\}_{k \in \mathcal{C}}$. By assumption b) the map $\mathrm{Map}_{\mathcal{C}}(c, f)$ is an equivalence for all $c \in \mathcal{C}$. So f is an equivalence itself by the Yoneda Lemma.

We first recall some preliminaries before we prove the implication *a)* $\Leftarrow$ *b)*. Because $\mathcal{C}$ is presentable, we can choose a regular cardinal κ so that the objects $\{c_k\}_{k \in K}$ are contained in the full subcategory $\mathcal{C}^{\kappa}$ of $\mathcal{C}$ spanned by κ-compact objects, see [KNP24, Lemma 2.1.16.]. Moreover, the category $\mathcal{C}^\kappa$ is small and closed under κ-small colimits, see [KNP24, Lemma 2.1.16.] and [Lur09, Theorem 5.5.1.1]. Let $\mathcal{C}_0$ be the smallest full subcategory of $\mathcal{C}^\kappa$ which contains the objects $\{c_k\}_{k\in K}$ and is closed under κ-small colimits in $\mathcal{C}^\kappa$. Because the category $\mathcal{C}^\kappa$ is small so is the category $\mathcal{C}_0$.

By [Lur09, Proposition 5.3.5.11] the unique κ-continuous functor $p : \mathrm{Ind}_\kappa(\mathcal{C}_0) \to \mathcal{C}$ extending $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is fully faithful. [Lur09, Corollary 5.3.5.4] identifies objects of the full subcategory $\mathrm{Ind}_\kappa(\mathcal{C}_0) \subseteq \mathcal{P}(\mathcal{C}_0)$ with κ-small limit preserving functors $\mathcal{C}_0^{\mathrm{op}} \to \mathcal{S}$. Consequently, the restricted Yoneda embedding

$$\mathcal{C} \to \mathcal{P}(\mathcal{C}_0), \qquad c \mapsto \mathrm{Map}_{\mathcal{C}}(-, c)$$

factors through a functor $y : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C}_0)$. We show that y is right adjoint to the functor $p : \mathrm{Ind}_\kappa(\mathcal{C}_0) \to \mathcal{C}$. To this end, we fix X in $\mathcal{C}$. It suffices to show that the two functors $\mathrm{Ind}_\kappa(\mathcal{C}_0) \to \mathcal{S}^{\mathrm{op}}$ given by

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}_0)}(-, y(X)) \qquad \text{and} \qquad \mathrm{Map}_{\mathcal{C}}(p(-), X)$$

are equivalent. As both of these functors preserve κ-filtered colimits, it suffices to construct an equivalence after precomposition with the functor $\mathcal{C}_0 \to \mathrm{Ind}_\kappa(\mathcal{C}_0)$. To this end we employ the Yoneda Lemma to identify the following functors $\mathcal{C}_0^{\mathrm{op}} \to \mathcal{S}$:

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C}_0)}(y(-), y(X)) = \mathrm{Map}_{\mathcal{P}(\mathcal{C}_0)}(y(-), y(X)) \simeq y(X) = \mathrm{Map}_{\mathcal{C}}(-, X).$$

The upshot of this preliminary discussion is that the restricted Yoneda embedding y induces a right adjoint of the fully faithful functor p. If condition a) is satisfied then the restricted Yoneda embedding y is a conservative functor. It follows from the triangle identities that p is an equivalence of categories. In particular, any object of $\mathcal{C}$ can be written as a κ-filtered colimit of a functor taking values in $\mathcal{C}_0$. Condition b) follows now by unraveling the definition of $\mathcal{C}_0$. □

Examples 2.2. Here are some examples of generating sets of presentable categories:

- The category of n-truncated ∞-groupoids is generated under colimits by the terminal object.
- The category of left modules over an $\mathbb{E}_1$-ring R is generated under colimits by desuspensions of the unit $\{R[-n]\}_{n\in\mathbb{N}_0}$.

3. Specialized Quillen's Theorem A

Lemma 3.1. Let $\mathcal{C}$ be a cocomplete category κ-compactly generated and $c \in \mathcal{C}$ an object. The functor $\mathrm{Map}_{\mathcal{C}}(c, -)$ corepresented by c admits a left adjoint

$$- \otimes c : \mathcal{S} \to \mathcal{C}, \quad X \mapsto \mathrm{colim}\left(X \to * \xrightarrow{c} \mathcal{C}\right)$$

sending an ∞-groupoid to the constant colimit indexed by X.

Proof.

$$\mathrm{Map}_{\mathcal{C}}(X \otimes c, d) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_X\, c, d) \simeq \lim_X \mathrm{Map}(c, d) \simeq \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}(c, d))$$

□

Lemma 3.2. Let $\mathcal{C}$ be a cocomplete category, J a small category and $j \in J$ an object. The evaluation at j functor $\mathrm{ev}_j : \mathrm{Fun}(J, \mathcal{C}) \to \mathcal{C}$ admits a left adjoint $j_! : \mathcal{C} \to \mathrm{Fun}(J, \mathcal{C})$. For any $c \in \mathcal{C}$ the functor $j_!(c)$ is equivalent to the composite

$$J \xrightarrow{\mathrm{Map}_J(j,-)} \mathcal{S} \xrightarrow{-\otimes c} \mathcal{C}\,. \tag{1}$$

Proof. The evaluation functor is given by precomposition along $j : \{j\} \to J$. Left Kan extension provides a left adjoint $j_! := \mathrm{Lan}_j$ to precomposition by $\{j\} \to J$. By the pointwise formula for left Kan extension we obtain an equivalence

$$j_!(c)(i) \simeq \operatorname*{colim}_{\{j\}\times_J J_{/i}} \left(\{j\} \times_J J_{/i} \to \{j\} \xrightarrow{c} \mathcal{C}\right) \simeq \operatorname*{colim}_{\mathrm{Map}_J(j,i)} \mathrm{const}(c) = \mathrm{Map}_J(j,i) \otimes c,$$

naturally in i. □

Lemma 3.3. Suppose a small set of objects $\{c_k\}_{k\in K}$ generates a presentable category $\mathcal{C}$ under small colimits. Let J be small category. Then, the functor category $\mathrm{Fun}(J,\mathcal{C})$ is presentable. Moreover, the small set

$$\left\{j_!(c_k) = \mathrm{Map}_j(j,-) \otimes c_k : \; j \in J, k \in K\right\}$$

generates the functor category $\mathrm{Fun}(J,\mathcal{C})$ under small colimits $\mathrm{Fun}(J,\mathcal{C})$.

Proof. The category $\mathrm{Fun}(J,\mathcal{C})$ is presentable by [Lur09, Proposition 5.4.4.3.]. Let f be a morphism in $\mathrm{Fun}(J,\mathcal{C})$ such that the induced morphism of mapping spaces $\mathrm{Map}(j_!(c_k), f)$ is an equivalence for all $j \in J$ and $k \in K$. Applying the adjunction from Lemma 3.2, we conclude that the morphism $\mathrm{Map}(c_k, \mathrm{ev}_j(f))$ is an equivalence. Because the set of objects $\{c_k\}_{k\in K}$ generates $\mathcal{C}$, the map $\mathrm{ev}_j(f)$ is an equivalence in $\mathcal{C}$. By the pointwise criterion for equivalences in functor categories, the morphism f is an equivalence itself. □

Lemma 3.4. Let $f : I \to J$ be a functor between small categories. Let $\mathcal{C}$ be a complete category and $j \in J$ and $c \in \mathcal{C}$ be objects. Then, the canonical map

$$\mathrm{colim}_I (j_!(c) \circ f) \to \mathrm{colim}_J \, j_!(c)$$

is homotopic to the canonical map

$$|I_{j/}| \otimes c \to c.$$

Proof. In the factorization (1) of $j_!(c)$ the latter functor preserves colimits by Lemma 3.1. Thus, it suffices to show that the canonical map of mapping spaces

$$\mathrm{colim}_I \, \mathrm{Map}_J(j,-) \to \mathrm{colim}_J \, \mathrm{Map}_J(j,-) \tag{2}$$

is equivalent to $|I_{j/}| \to *$. The colimit of an ∞-groupoid valued functor is computed by inverting the morphisms of the unstraightening. The unstraightening of $\mathrm{Map}_J(j,-) : J \to \mathcal{S}$ is the forgetful functor $J_{j/} \to J$. The unstraightening of the composite

$$I \xrightarrow{f} J \xrightarrow{\mathrm{Map}_J(j,-)} \mathcal{S}$$

is computed as the pullback of $J_{j/}$ along f. This identifies the map in Equation (2) with $|I \times_J J_{j/}| \to |J_{j/}|$. Finally, $|J_{j/}|$ is contractible because $J_{j/}$ has an initial object. □

Theorem 3.5 (Specialized Quillen's Theorem A)**.** *Let $f : I \to J$ be a functor of small categories and $\mathcal{C}$ a presentable category. Choose any small set of objects $\{c_k\}_{k\in K}$ which generate $\mathcal{C}$ under small colimits. The following are equivalent:*

- *a) For any functor $F : J \to \mathcal{C}$ the canonical map $\mathrm{colim}_I F \circ f \to \mathrm{colim}_J F$ is an equivalence in $\mathcal{C}$.*
- *b) For any $j \in J$ and $k \in K$ the map $|I_{j/}| \otimes c_k \to c_k$ is an equivalence in $\mathcal{C}$.*
- *c) For any $j \in J$ and $k \in K$ the canonical map $\mathrm{colim}_I \, j_!(c_k) \circ f \to \mathrm{colim}_J \, j_!(c_k)$ is an equivalence in $\mathcal{C}$.*

Proof. The implications b) $\iff$ c) are immediate from Lemma 3.4. As c) is a special case of a), we are left to proving that a) implies c). Let $\mathrm{Fun}(J,C)_0$ be the full subcategory of $\mathrm{Fun}(J,C)$ spanned by those functors $F : J \to \mathcal{C}$ for which the canonical map $\mathrm{colim}_I\, F \circ f \to \mathrm{colim}_J\, F$ is an equivalence. We check that $\mathcal{C}_0$ is closed under small colimits in $\mathcal{C}$:

Let $p_J : \mathrm{Fun}(J,\mathcal{C}) \to \mathcal{C}$ denote the J-indexed colimit functor, let $f^* : \mathrm{Fun}(J,\mathcal{C}) \to \mathrm{Fun}(I,\mathcal{C})$ denote precomposition by f and let $p_I : \mathrm{Fun}(J,\mathcal{C}) \to \mathcal{C}$ denote the I-indexed colimit functor. Let $\eta : p_I \circ f^* \Rightarrow p_J$ denote the canonical natural transformation. By definition, we have $F \in \mathrm{Fun}(J,C)_0$ if and only if $F(\eta)$ is an equivalence. Let T be a small category and

$$T \to \mathrm{Fun}(J,C), \qquad t \to F_t$$

a diagram which factors through $\mathrm{Fun}(J,C)_0$. By assumption the top horizontal arrow in the following commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_T\, p_J(F_t) \arrow[r, "\mathrm{colim}_T\, \eta(F_t)"] \arrow[d, "\simeq"'] & \mathrm{colim}_T(p_I \circ f^*)(F_t) \arrow[d, "\simeq"] \\
p_J(\mathrm{colim}_T\, F_t) \arrow[r, "\eta(\mathrm{colim}_T\, F_t)"] & (p_I \circ f^*)(\mathrm{colim}_T\, F_t)
\end{tikzcd}$$

is an equivalence. As the functors p_I, p_J and f^* preserve small colimits, the vertical arrows are equivalences, as well. We conclude that the colimit $\mathrm{colim}_T\, F_t$ is contained in $\mathrm{Fun}(J,C)_0$, too.

By Lemma 3.3 and Proposition 2.1 the category $\mathrm{Fun}(J,\mathcal{C})$ itself is the smallest full subcategory of $\mathrm{Fun}(J,\mathcal{C})$ which is closed under small colimits and contains the object $j_!(c_k)$ for all $j \in J$ and $k \in K$. If assumption c) holds then $\mathrm{Fun}(J,C)_0$ is another such subcategory. We conclude that $\mathrm{Fun}(J,C) \subseteq \mathrm{Fun}(J,C)_0$ so that a) holds. □

4. $\mathcal{R}$-linear Quillen's Theorem A

Theorem 4.1 ($\mathcal{R}$-linear Quillen's Theorem A)**.** *Let $(\mathcal{R}, \otimes, \mathbb{1}_R)$ be a presentably monoidal category and $f : I \to J$ a functor between small categories. Then the following are equivalent:*

- *a) For any presentably $\mathcal{R}$-linear category $\mathcal{C}$ and any functor $F : J \to \mathcal{C}$, the canonical map $\mathrm{colim}_I(F \circ f) \to \mathrm{colim}_J\, F$ is an equivalence.*
- *b) For any $j \in J$, the canonical map $|I_{j/}| \otimes \mathbb{1}_R \to \mathbb{1}_R$ is an equivalence in $\mathcal{R}$.*

Proof. As $\mathcal{R}$ itself is presentably $\mathcal{R}$-linear the implication $a) \Rightarrow b)$ follows from Theorem 3.5 applied to $\mathcal{C} = \mathcal{R}$ and $c_k = \mathbb{1}_{\mathcal{R}}$. Let us assume b) holds and prove a). Let $\mathcal{C}$ be a presentably $\mathcal{R}$-linear category. By Theorem 3.5 it suffices to show that for all $c \in \mathcal{C}$ and $j \in J$ the map $|I_{j/}| \otimes c \to c$ is an equivalence in $\mathcal{C}$. Because $\mathcal{C}$ is presentably $\mathcal{R}$-linear there exists some colimit preserving functor $- \otimes c : \mathcal{R} \to \mathcal{C}$ and a natural transformation making the diagram

$$\begin{tikzcd}
 & \mathcal{R} \arrow[dl, dashed, "\otimes_{\mathcal{R}} c"'] & \\
\mathcal{C} & & \mathcal{S} \arrow[ul, "\otimes \mathbb{1}_{\mathcal{R}}"'] \arrow[ll, "\otimes c"]
\end{tikzcd}$$

commute. Indeed, this follows from the definition of an presentably $\mathcal{R}$-linear category by passage to left adjoints, see Lemma 3.1. We apply the functor $- \otimes_{\mathcal{R}} c : \mathcal{R} \to \mathcal{C}$ to the equivalence $|I_{j/}| \otimes \mathbb{1}_R \to \mathbb{1}_R$ to see that the canonical map

$$|I_{j/}| \otimes c \simeq \big(|I_{j/}| \otimes \mathbb{1}_R\big) \otimes_{\mathcal{R}} c \xrightarrow{\simeq} \mathbb{1}_R \otimes_{\mathcal{R}} c \simeq c$$

is an equivalence as well. □

References

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[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009. 3, 4